

# Estimation and Confidence Intervals

## Chapter 9



# GOALS

- Define a point estimate.
- Define level of confidence.
- Construct a confidence interval for the population mean when the population standard deviation is known.
- Construct a confidence interval for a population mean when the population standard deviation is unknown.
- Construct a confidence interval for a population proportion.
- Determine the sample size for attribute and variable sampling.

# Point and Interval Estimates

- A point estimate is the statistic, computed from sample information, which is used to estimate the population parameter.
- A confidence interval estimate is a range of values constructed from sample data so that the population parameter is likely to occur within that range at a specified probability. The specified probability is called the level of confidence.

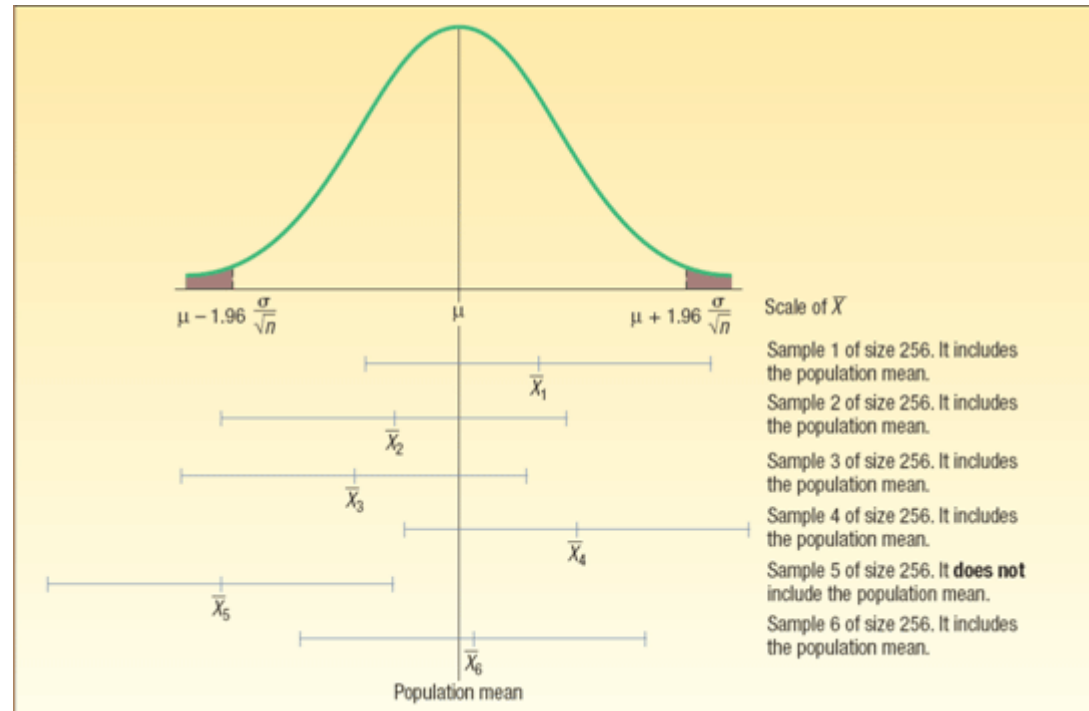
## Factors Affecting Confidence Interval Estimates

The factors that determine the width of a confidence interval are:

1. The sample size,  $n$ .
2. The variability in the population, usually  $\sigma$  estimated by  $s$ .
3. The desired level of confidence.

# Interval Estimates - Interpretation

For a 95% confidence interval about 95% of the similarly constructed intervals will contain the parameter being estimated. Also 95% of the sample means for a specified sample size will lie within 1.96 standard deviations of the hypothesized population



# Characteristics of the $t$ -distribution

1. It is, like the  $z$  distribution, a continuous distribution.
2. It is, like the  $z$  distribution, bell-shaped and symmetrical.
3. There is not one  $t$  distribution, but rather a family of  $t$  distributions. All  $t$  distributions have a mean of 0, but their standard deviations differ according to the sample size,  $n$ .
4. The  $t$  distribution is more spread out and flatter at the center than the standard normal distribution. As the sample size increases, however, the  $t$  distribution approaches the standard normal distribution,

# Comparing the z and t Distributions when $n$ is small

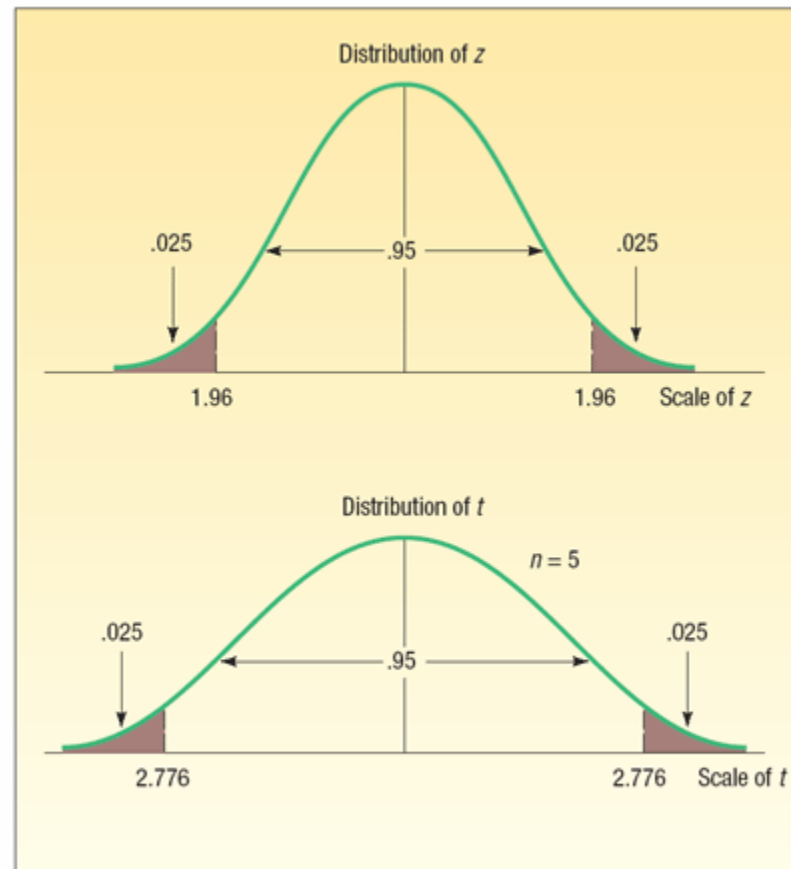


CHART 9-2 Values of  $z$  and  $t$  for the 95 Percent Level of Confidence

# Confidence Interval Estimates for the Mean

## Use $Z$ -distribution

If the population standard deviation is known or the sample is greater than 30.

$$\bar{X} \pm z \frac{\sigma}{\sqrt{n}}$$

## Use $t$ -distribution

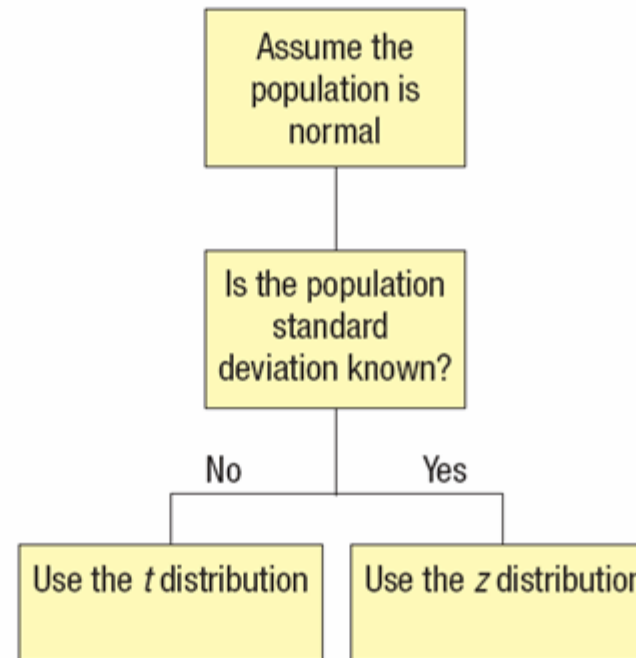
If the population standard deviation is unknown and the sample is less than 30.

$$\bar{X} \pm t \frac{s}{\sqrt{n}}$$



# When to Use the $z$ or $t$ Distribution for Confidence Interval Computation

## Estimation and Confidence Intervals



**CHART 9–3** Determining When to Use the  $z$  Distribution or the  $t$  Distribution

# Confidence Interval for the Mean – Example using the t-distribution

A tire manufacturer wishes to investigate the tread life of its tires. A sample of 10 tires driven 50,000 miles revealed a sample mean of 0.32 inch of tread remaining with a standard deviation of 0.09 inch. Construct a 95 percent confidence interval for the population mean. Would it be reasonable for the manufacturer to conclude that after 50,000 miles the population mean amount of tread remaining is 0.30 inches?

Given in the problem :

$$n = 10$$

$$\bar{x} = 0.32$$

$$s = 0.09$$

Compute the C.I. using the t - dist. (since  $\sigma$  is unknown)

$$\bar{X} \pm t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$$

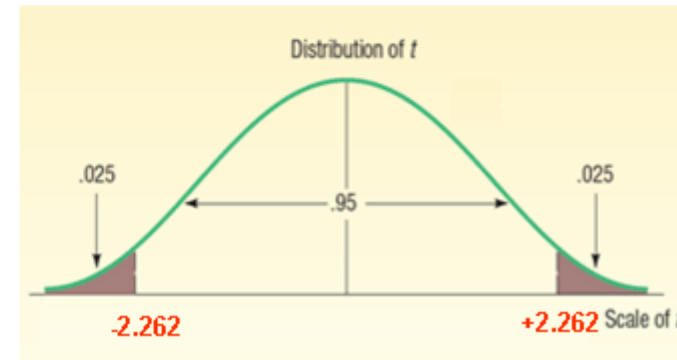
# Student's t-distribution Table

Compute the C.I.  
using the t - dist. (since  $\sigma$  is unknown)

$$\begin{aligned} & \bar{X} \pm t_{\alpha/2, n-1} \frac{s}{\sqrt{n}} \\ &= \bar{X} \pm t_{.05/2, 10-1} \frac{s}{\sqrt{n}} \\ &= 0.32 \pm t_{.025, 9} \frac{0.09}{\sqrt{10}} \\ &= 0.32 \pm 2.262 \frac{0.09}{\sqrt{10}} \\ &= 0.32 \pm 0.064 \\ &= (0.256, 0.384) \end{aligned}$$

Conclude : the manufacturer can be reasonably sure (95% confident) that the mean remaining tread depth is between 0.256 and 0.384 inches.

df	Confidence Intervals				
	80%	90%	95%	98%	99%
	Level of Significance for One-Tailed Test				
	0.100	0.050	0.025	0.010	0.005
Level of Significance for Two-Tailed Test					
	0.20	0.10	0.05	0.02	0.01
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169



# Confidence Interval Estimates for the Mean – Using Minitab



The manager of the Inlet Square Mall, near Ft. Myers, Florida, wants to estimate the mean amount spent per shopping visit by customers. A sample of 20 customers reveals the following amounts spent.

\$48.16	\$42.22	\$46.82	\$51.45	\$23.78	\$41.86	\$54.86
37.92	52.64	48.59	50.82	46.94	61.83	61.69
49.17	61.46	51.35	52.68	58.84	43.88	

# Confidence Interval Estimates for the Mean – By Formula

Compute the C.I.

using the t - dist. (since  $\sigma$  is unknown)

$$\begin{aligned} & \bar{X} \pm t_{\alpha/2, n-1} \frac{s}{\sqrt{n}} \\ &= \bar{X} \pm t_{.05/2, 20-1} \frac{s}{\sqrt{n}} \\ &= 49.35 \pm t_{.025, 19} \frac{9.01}{\sqrt{20}} \\ &= 49.35 \pm 2.093 \frac{9.01}{\sqrt{20}} \\ &= 49.35 \pm 4.22 \end{aligned}$$

The endpoints of the confidence interval are \$45.13 and \$53.57

Conclude : It is reasonable that the population mean could be \$50.

The value of \$60 is not in the confidence interval. Hence, we conclude that the population mean is unlikely to be \$60.

# Confidence Interval Estimates for the Mean – Using Minitab

The screenshot displays the Minitab software interface. On the left, a worksheet titled 'Worksheet 1' contains a column of data labeled 'Amount' with 20 rows of values. On the right, a 'Session' window shows the following output:

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Welcome to Minitab, press F1 for help.

**Descriptive Statistics: Amount**

Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3	Maximum
Amount	20	0	49.35	2.02	9.01	23.78	44.62	50.00	54.32	61.83

**One-Sample T: Amount**

Variable	N	Mean	StDev	SE Mean	95% CI
Amount	20	49.3480	9.0121	2.0152	(45.1302, 53.5658)

The 95% CI value (45.1302, 53.5658) is circled in red in the original image.

Current Worksheet: Worksheet 1

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# Using the Normal Distribution to Approximate the Binomial Distribution

To develop a confidence interval for a proportion, we need to meet the following assumptions.

1. The binomial conditions, discussed in Chapter 6, have been met. Briefly, these conditions are:
  - a. The sample data is the result of counts.
  - b. There are only two possible outcomes.
  - c. The probability of a success remains the same from one trial to the next.
  - d. The trials are independent. This means the outcome on one trial does not affect the outcome on another.
2. The values  $n\pi$  and  $n(1-\pi)$  should both be greater than or equal to 5. This condition allows us to invoke the central limit theorem and employ the standard normal distribution, that is,  $z$ , to complete a confidence interval.



## Confidence Interval for a Population Proportion

The confidence interval for a population proportion is estimated by:

$$p \pm z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}}$$

# Confidence Interval for a Population Proportion – Example

The union representing the Bottle Blowers of America (BBA) is considering a proposal to merge with the Teamsters Union. According to BBA union bylaws, at least three-fourths of the union membership must approve any merger. A random sample of 2,000 current BBA members reveals 1,600 plan to vote for the merger proposal. What is the estimate of the population proportion?

Develop a 95 percent confidence interval for the population proportion. Basing your decision on this sample information, can you conclude that the necessary proportion of BBA members favor the merger? Why?

First, compute the sample proportion :

$$p = \frac{x}{n} = \frac{1,600}{2000} = 0.80$$

Compute the 95% C.I.

$$\begin{aligned} \text{C.I.} &= p \pm z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} \\ &= 0.80 \pm 1.96 \sqrt{\frac{.80(1-.80)}{2,000}} = .80 \pm .018 \\ &= (0.782, 0.818) \end{aligned}$$

Conclude : The merger proposal will likely pass because the interval estimate includes values greater than 75 percent of the union membership .

# Finite-Population Correction Factor

- A population that has a fixed upper bound is said to be finite.
- For a finite population, where the total number of objects is  $N$  and the size of the sample is  $n$ , the following adjustment is made to the standard errors of the sample means and the proportion:
- However, if  $n/N < .05$ , the finite-population correction factor may be ignored.

Standard Error of the  
Sample Mean

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}$$

Standard Error of the  
Sample Proportion

$$\sigma_p = \sqrt{\frac{p(1-p)}{n}} \sqrt{\frac{N-n}{N-1}}$$

# Effects on FPC when $n/N$ Changes

Finite-Population Correction Factor for Selected Samples When the Population Is 1,000

Sample Size	Fraction of Population	Correction Factor
10	.010	.9955
25	.025	.9879
50	.050	.9752
100	.100	.9492
200	.200	.8949
500	.500	.7075

Observe that FPC approaches 1 when  $n/N$  becomes smaller

# Confidence Interval Formulas for Estimating Means and Proportions with Finite Population Correction

C.I. for the Mean ( $\mu$ )

$$\bar{X} \pm z \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}$$

C.I. for the Mean ( $\mu$ )

$$\bar{X} \pm t \frac{s}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}$$

C.I. for the Proportion ( $\pi$ )

$$p \pm z \sqrt{\frac{p(1-p)}{n}} \sqrt{\frac{N-n}{N-1}}$$

# CI For Mean with FPC - Example

There are 250 families in Scandia, Pennsylvania. A random sample of 40 of these families revealed the mean annual church contribution was \$450 and the standard deviation of this was \$75.

Develop a 90 percent confidence interval for the population mean.

Interpret the confidence interval.

Given in Problem:

N – 250

n – 40

s - \$75

Since  $n/N = 40/250 = 0.16$ , the finite population correction factor must be used.

The population standard deviation is not known therefore use the t-distribution (may use the z-dist since  $n > 30$ )

Use the formula below to compute the confidence interval:

$$\bar{X} \pm t \frac{s}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}$$

# CI For Mean with FPC - Example

$$\begin{aligned} & \bar{X} \pm t \frac{s}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}} \\ &= \$450 \pm t_{.10,40-1} \frac{\$75}{\sqrt{40}} \sqrt{\frac{250-40}{250-1}} \\ &= \$450 \pm 1.685 \frac{\$75}{\sqrt{40}} \sqrt{\frac{250-40}{250-1}} \\ &= \$450 \pm \$19.98 \sqrt{.8434} \\ &= \$450 \pm \$18.35 \\ &= (\$431.65, \$468.35) \end{aligned}$$

It is likely that the population mean is more than \$431.65 but less than \$468.35. To put it another way, could the population mean be \$445? Yes, but it is not likely that it is \$425 because the value \$445 is within the confidence interval and \$425 is not within the confidence interval.

# Selecting a Sample Size

There are 3 factors that determine the size of a sample, none of which has any direct relationship to the size of the population. They are:

- **The degree of confidence selected.**
- **The maximum allowable error.**
- **The variation in the population.**



# Sample Size Determination for a Variable

- To find the sample size for a variable:

$$n = \left( \frac{z \cdot s}{E} \right)^2$$

where :

$E$  - the allowable error

$z$  - the  $z$  - value corresponding to the selected level of confidence

$s$  - the sample deviation (from pilot sample)

# Sample Size Determination for a Variable-Example

A student in public administration wants to determine the mean amount members of city councils in large cities earn per month as remuneration for being a council member. The error in estimating the mean is to be less than \$100 with a 95 percent level of confidence. The student found a report by the Department of Labor that estimated the standard deviation to be \$1,000. What is the required sample size?

Given in the problem:

- $E$ , the maximum allowable error, is \$100
- The value of  $z$  for a 95 percent level of confidence is 1.96,
- The estimate of the standard deviation is \$1,000.

$$\begin{aligned}n &= \left( \frac{z \cdot s}{E} \right)^2 \\&= \left( \frac{1.96 \cdot \$1,000}{\$100} \right)^2 \\&= (19.6)^2 \\&= 384.16 \\&= 385\end{aligned}$$

# Sample Size Determination for a Variable- Another Example

A consumer group would like to estimate the mean monthly electricity charge for a single family house in July within \$5 using a 99 percent level of confidence. Based on similar studies the standard deviation is estimated to be \$20.00. How large a sample is required?

$$n = \left( \frac{(2.58)(20)}{5} \right)^2 = 107$$

# Sample Size for Proportions

- The formula for determining the sample size in the case of a proportion is:

$$n = p(1 - p) \left( \frac{Z}{E} \right)^2$$

where :

$p$  is estimate from a pilot study or some source,  
otherwise, 0.50 is used

$z$  - the  $z$  - value for the desired confidence level

$E$  - the maximum allowable error

# Another Example

The American Kennel Club wanted to estimate the proportion of children that have a dog as a pet. If the club wanted the estimate to be within 3% of the population proportion, how many children would they need to contact? Assume a 95% level of confidence and that the club estimated that 30% of the children have a dog as a pet.

$$n = (.30)(.70) \left( \frac{1.96}{.03} \right)^2 = 897$$

# Another Example

A study needs to estimate the proportion of cities that have private refuse collectors. The investigator wants the margin of error to be within .10 of the population proportion, the desired level of confidence is 90 percent, and no estimate is available for the population proportion. What is the required sample size?

$$n = (.5)(1 - .5) \left( \frac{1.65}{.10} \right)^2 = 68.0625$$

$$n = 69 \text{ cities}$$

End of Chapter 9

# One Sample Tests of Hypothesis

Chapter 10





# GOALS

- Define a hypothesis and hypothesis testing.
- Describe the five-step hypothesis-testing procedure.
- Distinguish between a one-tailed and a two-tailed test of hypothesis.
- Conduct a test of hypothesis about a population mean.
- Conduct a test of hypothesis about a population proportion.
- Define Type I and Type II errors.
- Compute the probability of a Type II error.

# What is a Hypothesis?

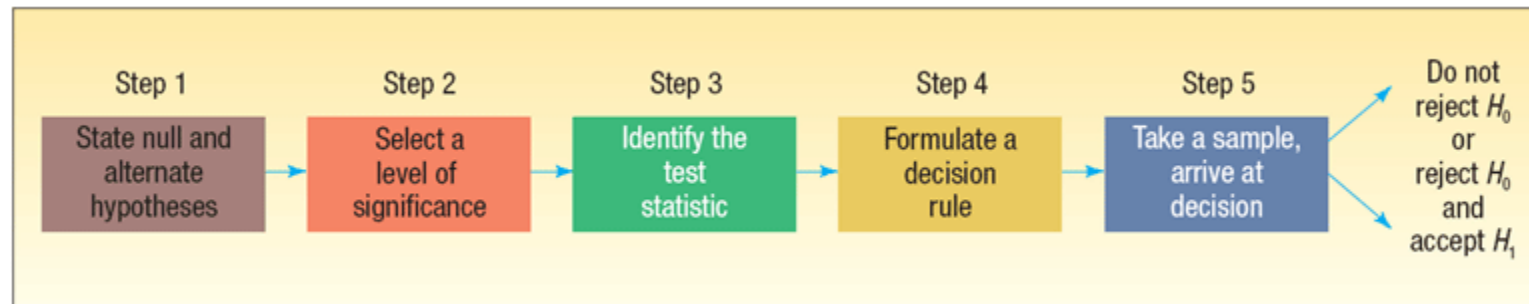
A **Hypothesis** is a statement about the value of a population parameter developed for the purpose of testing. Examples of hypothesis made about a population parameter are:

- The mean monthly income for systems analysts is \$3,625.
- Twenty percent of all customers at Bovine's Chop House return for another meal within a month.

# What is Hypothesis Testing?

**Hypothesis testing** is a procedure, based on sample evidence and probability theory, used to determine whether the hypothesis is a reasonable statement and should not be rejected, or is unreasonable and should be rejected.

# Hypothesis Testing Steps



## Important Things to Remember about $H_0$ and $H_1$

- $H_0$ : null hypothesis and  $H_1$ : alternate hypothesis
- $H_0$  and  $H_1$  are mutually exclusive and collectively exhaustive
- $H_0$  is always presumed to be true
- $H_1$  has the burden of proof
- A random sample ( $n$ ) is used to “*reject  $H_0$* ”
- If we conclude 'do not reject  $H_0$ ', this does not necessarily mean that the null hypothesis is true, it only suggests that there is not sufficient evidence to reject  $H_0$ ; rejecting the null hypothesis then, suggests that the alternative hypothesis may be true.
- Equality is always part of  $H_0$  (e.g. “=”, “ $\geq$ ”, “ $\leq$ ”).
- “ $\neq$ ”, “ $<$ ” and “ $>$ ” always part of  $H_1$

# How to Set Up a Claim as Hypothesis

- In actual practice, the status quo is set up as  $H_0$
- If the claim is “boastful” the claim is set up as  $H_1$  (we apply the Missouri rule – “show me”).  
Remember,  $H_1$  has the burden of proof
- In problem solving, look for **key words** and convert them into symbols. Some key words include: “**improved, better than, as effective as, different from, has changed**, etc.”

# Left-tail or Right-tail Test?

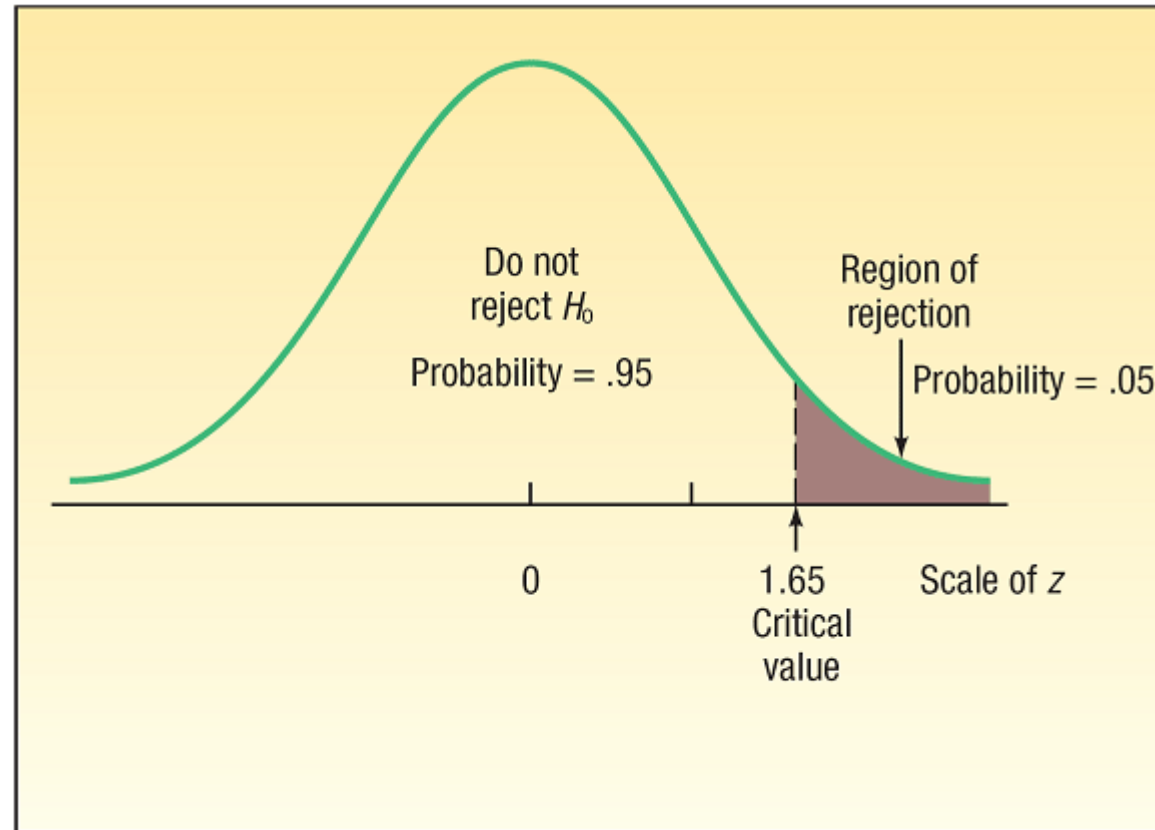
- The direction of the test involving claims that use the words “*has improved*”, “*is better than*”, and the like will depend upon the variable being measured.
- For instance, if the variable involves *time for a certain medication to take effect*, the words “better” “improve” or more effective” are translated as “<” (less than, i.e. faster relief).
- On the other hand, if the variable refers to a *test score*, then the words “better” “improve” or more effective” are translated as “>” (greater than, i.e. higher test scores)

Keywords	Inequality Symbol	Part of:
<i>Larger (or more) than</i>	$>$	$H_1$
<i>Smaller (or less)</i>	$<$	$H_1$
<i>No more than</i>	$\leq$	$H_0$
<i>At least</i>	$\geq$	$H_0$
<i>Has increased</i>	$>$	$H_1$
<i>Is there difference?</i>	$\neq$	$H_1$
<i>Has not changed</i>	$=$	$H_0$
<i>Has “improved”, “is better than”, “is more effective”</i>	See right	$H_1$

Null Hypothesis	Researcher	
	Does Not Reject $H_0$	Rejects $H_0$
$H_0$ is true	Correct decision	Type I error
$H_0$ is false	Type II error	Correct decision

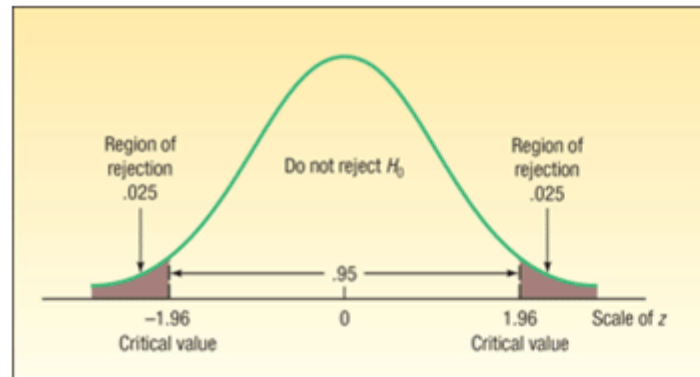


# Parts of a Distribution in Hypothesis Testing

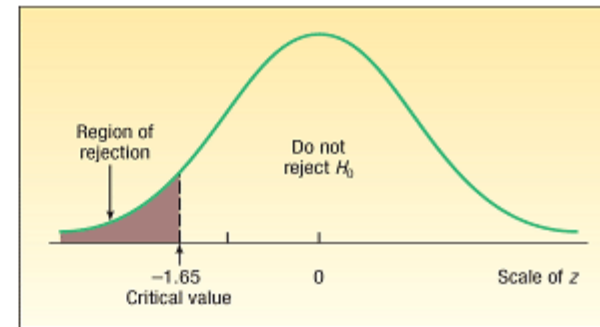


# One-tail vs. Two-tail Test

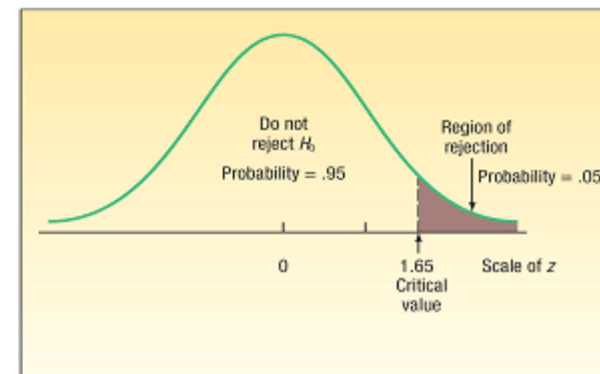
## Two-tail or Non-directional Test



## One-tail, Left Tail Test



## One-tail, Right Tail Test



# Hypothesis Setups for Testing a Mean ( $\mu$ )

$H_0: \mu = \text{value}$   
 $H_1: \mu \neq \text{value}$

Reject  $H_0$  if:  
 $|Z| > Z_{\alpha/2}$   
 $|t| > t_{\alpha/2, n-1}$

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$H_0: \mu \geq \text{value}$   
 $H_1: \mu < \text{value}$

Reject  $H_0$  if:  
 $Z < -Z_{\alpha}$   
 $t < -t_{\alpha, n-1}$

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

$H_0: \mu \leq \text{value}$   
 $H_1: \mu > \text{value}$

Reject  $H_0$  if:  
 $Z > Z_{\alpha}$   
 $t > t_{\alpha, n-1}$

# Hypothesis Setups for Testing a Proportion ( $\pi$ )

$H_0: \pi = \text{value}$   
 $H_1: \pi \neq \text{value}$   
Reject  $H_0$  if:  
 $|Z| > Z_{\alpha/2}$

$H_0: \pi \geq \text{value}$   
 $H_1: \pi < \text{value}$   
Reject  $H_0$  if:  
 $Z < -Z_{\alpha}$

$H_0: \pi \leq \text{value}$   
 $H_1: \pi > \text{value}$   
Reject  $H_0$  if:  
 $Z > Z_{\alpha}$

$$z = \frac{p - \pi}{\sqrt{\frac{\pi(1 - \pi)}{n}}}$$

# Testing for a Population Mean with a Known Population Standard Deviation- Example

Jamestown Steel Company manufactures and assembles desks and other office equipment at several plants in western New York State. The weekly production of the Model A325 desk at the Fredonia Plant follows the normal probability distribution with a mean of 200 and a standard deviation of 16. Recently, because of market expansion, new production methods have been introduced and new employees hired. The vice president of manufacturing would like to investigate whether there has been a change in the weekly production of the Model A325 desk.



## Testing for a Population Mean with a Known Population Standard Deviation- Example

**Step 1: State the null hypothesis and the alternate hypothesis.**

$$H_0: \mu = 200$$

$$H_1: \mu \neq 200$$

**(note: keyword in the problem “has changed”)**

**Step 2: Select the level of significance.**

$$\alpha = 0.01 \text{ as stated in the problem}$$

**Step 3: Select the test statistic.**

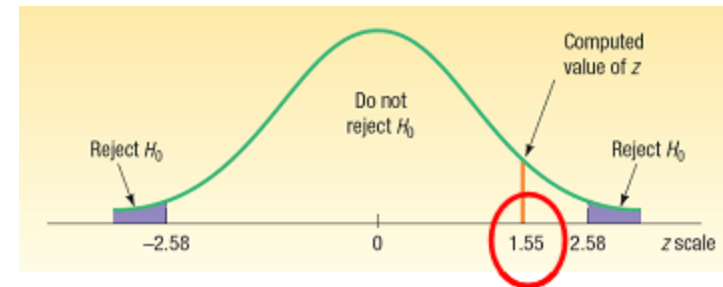
Use Z-distribution since  $\sigma$  is known

# Testing for a Population Mean with a Known Population Standard Deviation- Example

**Step 4: Formulate the decision rule.**

**Reject  $H_0$  if  $|Z| > Z_{\alpha/2}$**

$$\begin{aligned} |Z| &> Z_{\alpha/2} \\ \left| \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \right| &> Z_{\alpha/2} \\ \left| \frac{203.5 - 200}{16 / \sqrt{50}} \right| &> Z_{.01/2} \\ 1.55 &\text{ is not } > 2.58 \end{aligned}$$



**Step 5: Make a decision and interpret the result.**

**Because 1.55 does not fall in the rejection region,  $H_0$  is not rejected. We conclude that the population mean is not different from 200. So we would report to the vice president of manufacturing that the sample evidence does not show that the production rate at the Fredonia Plant has changed from 200 per week.**

## Testing for a Population Mean with a Known Population Standard Deviation- Another Example

Suppose in the previous problem the vice president wants to know whether there has been an ***increase*** in the number of units assembled. To put it another way, can we conclude, because of the improved production methods, that the mean number of desks assembled in the last 50 weeks was more than 200?

Recall:  $\sigma=16$ ,  $n=200$ ,  $\alpha=.01$



# Testing for a Population Mean with a Known Population Standard Deviation- Example

**Step 1: State the null hypothesis and the alternate hypothesis.**

$$H_0: \mu \leq 200$$

$$H_1: \mu > 200$$

**(note: keyword in the problem “an increase”)**

**Step 2: Select the level of significance.**

$$\alpha = 0.01 \text{ as stated in the problem}$$

**Step 3: Select the test statistic.**

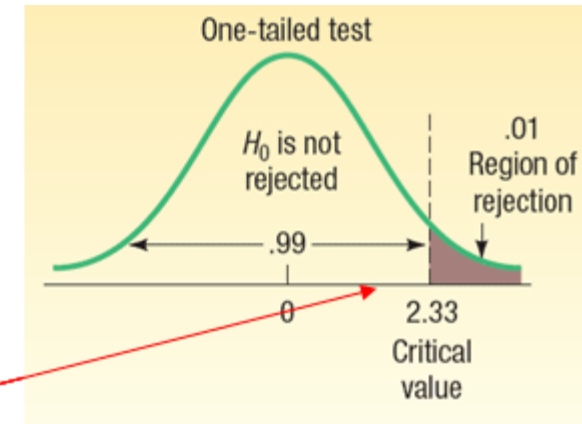
Use Z-distribution since  $\sigma$  is known

# Testing for a Population Mean with a Known Population Standard Deviation- Example

**Step 4: Formulate the decision rule.**

Reject  $H_0$  if  $Z > Z_\alpha$

$$\begin{aligned} Z &> Z_\alpha \\ \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} &> Z_\alpha \\ \frac{203.5 - 200}{16 / \sqrt{50}} &> Z_{.01} \\ 1.55 &\text{ is not } > 2.33 \end{aligned}$$



**Step 5: Make a decision and interpret the result.**

**Because 1.55 does not fall in the rejection region,  $H_0$  is not rejected.**

**We conclude that the average number of desks assembled in the last 50 weeks is not more than 200**

# Type of Errors in Hypothesis Testing

- Type I Error:
  - Defined as the probability of rejecting the null hypothesis when it is actually true.
  - This is denoted by the Greek letter “ $\alpha$ ”
  - Also known as the significance level of a test
- Type II Error:
  - Defined as the probability of “accepting” the null hypothesis when it is actually false.
  - This is denoted by the Greek letter “ $\beta$ ”

# $p$ -Value in Hypothesis Testing

- **p-VALUE** is the probability of observing a sample value as extreme as, or more extreme than, the value observed, given that the null hypothesis is true.
- In testing a hypothesis, we can also compare the  $p$ -value to with the significance level ( $\alpha$ ).
- If the  $p$ -value  $<$  significance level,  $H_0$  is rejected, else  $H_0$  is not rejected.

# $p$ -Value in Hypothesis Testing - Example

Recall the last problem where the hypothesis and decision rules were set up as:

$$H_0: \mu \leq 200$$

$$H_1: \mu > 200$$

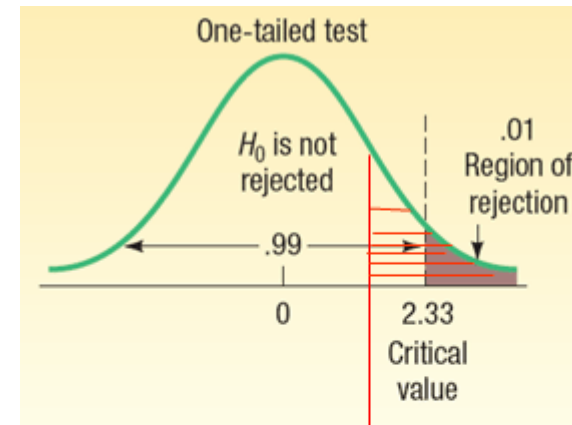
Reject  $H_0$  if  $Z > Z_\alpha$

where  $Z = 1.55$  and  $Z_\alpha = 2.33$

Reject  $H_0$  if  $p\text{-value} < \alpha$

0.0606 is  
not  $< 0.01$

Conclude: Fail to reject  $H_0$



1.55

$$P(Z > 1.55) = .5000 - .4394$$

$$P\text{-value} = .0606$$

What does it mean when p-value  $< \alpha$ ?

- (a) .10, we have some evidence that  $H_0$  is not true.
- (b) .05, we have strong evidence that  $H_0$  is not true.
- (c) .01, we have very strong evidence that  $H_0$  is not true.
- (d) .001, we have extremely strong evidence that  $H_0$  is not true.

## Testing for the Population Mean: Population Standard Deviation Unknown

- When the population standard deviation ( $\sigma$ ) is unknown, the sample standard deviation ( $s$ ) is used in its place
- The  $t$ -distribution is used as test statistic, which is computed using the formula:

**TESTING A MEAN,  $\sigma$  UNKNOWN**

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

**[10-2]**

with  $n - 1$  degrees of freedom, where:

$\bar{X}$  is the sample mean.

$\mu$  is the hypothesized population mean.

$s$  is the sample standard deviation.

$n$  is the number of observations in the sample.

# Testing for the Population Mean: Population Standard Deviation Unknown - Example

The McFarland Insurance Company Claims Department reports the mean cost to process a claim is \$60. An industry comparison showed this amount to be larger than most other insurance companies, so the company instituted cost-cutting measures. To evaluate the effect of the cost-cutting measures, the Supervisor of the Claims Department selected a random sample of 26 claims processed last month. The sample information is reported below.

At the .01 significance level is it reasonable a claim is now less than \$60?

\$45	\$49	\$62	\$40	\$43	\$61
48	53	67	63	78	64
48	54	51	56	63	69
58	51	58	59	56	57
38	76				



## Testing for a Population Mean with a Known Population Standard Deviation- Example

**Step 1: State the null hypothesis and the alternate hypothesis.**

$$H_0: \mu \geq \$60$$

$$H_1: \mu < \$60$$

(note: keyword in the problem “*now less than*”)

**Step 2: Select the level of significance.**

$$\alpha = 0.01 \text{ as stated in the problem}$$

**Step 3: Select the test statistic.**

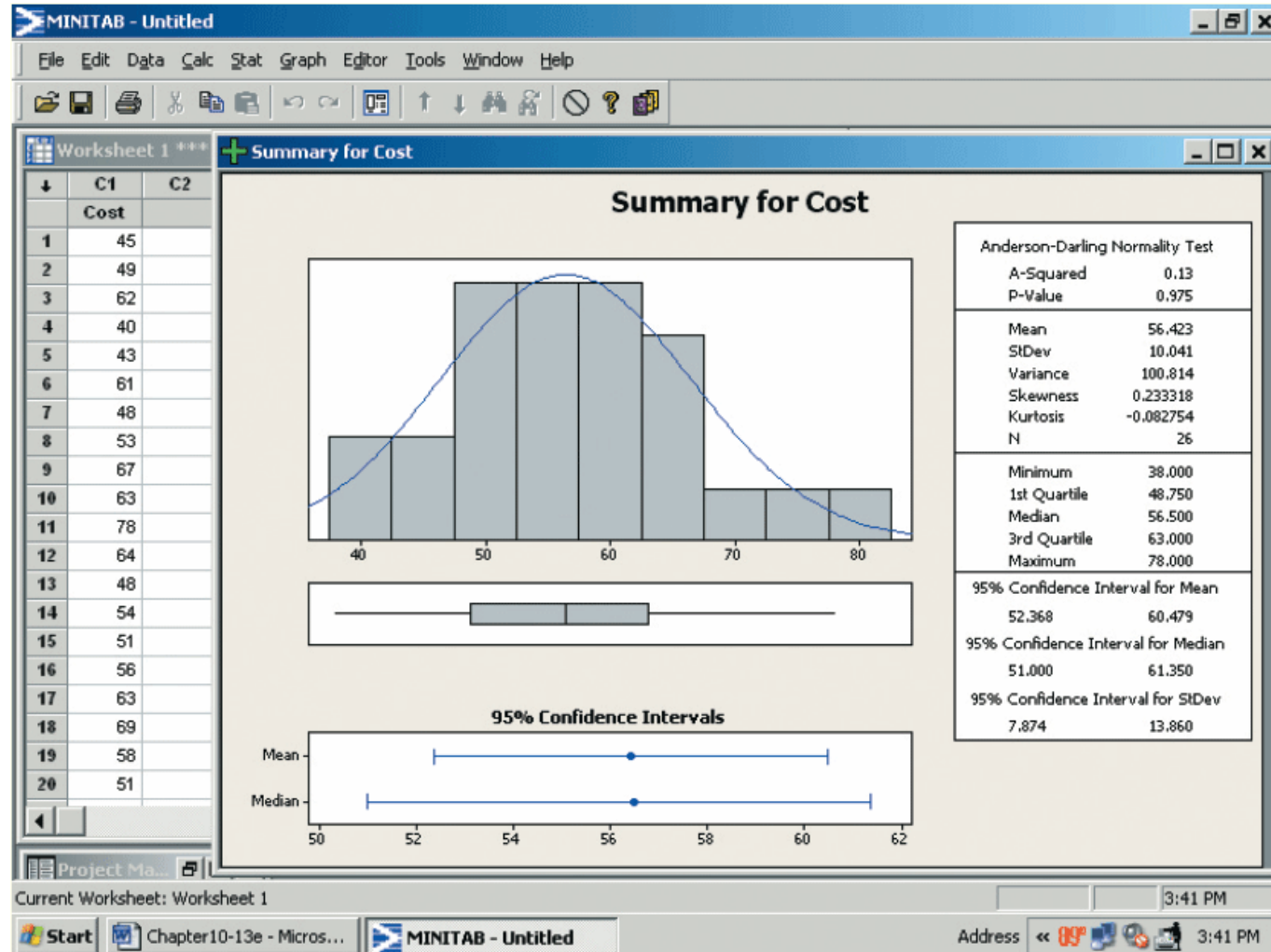
Use  $t$ -distribution since  $\sigma$  is unknown

# t-Distribution Table (portion)

**TABLE 10–1** A Portion of the *t* Distribution Table

Confidence Intervals						
	80%	90%	95%	98%	99%	99.9%
<i>df</i>	Level of Significance for One-Tailed Test, $\alpha$					
	0.100	0.050	0.025	0.010	0.005	0.0005
	Level of Significance for Two-Tailed Test, $\alpha$					
	0.20	0.10	0.05	0.02	0.01	0.001
⋮	⋮	⋮	⋮	⋮	⋮	⋮
21	1.323	1.721	2.080	2.518	2.831	3.819
22	1.321	1.717	2.074	2.508	2.819	3.792
23	1.319	1.714	2.069	2.500	2.807	3.768
24	1.318	1.711	2.064	2.492	2.797	3.745
25	1.316	1.708	2.060	2.485	2.787	3.725
26	1.315	1.706	2.056	2.479	2.779	3.707
27	1.314	1.703	2.052	2.473	2.771	3.690
28	1.313	1.701	2.048	2.467	2.763	3.674
29	1.311	1.699	2.045	2.462	2.756	3.659
30	1.310	1.697	2.042	2.457	2.750	3.646

# Testing for the Population Mean: Population Standard Deviation Unknown – Minitab Solution

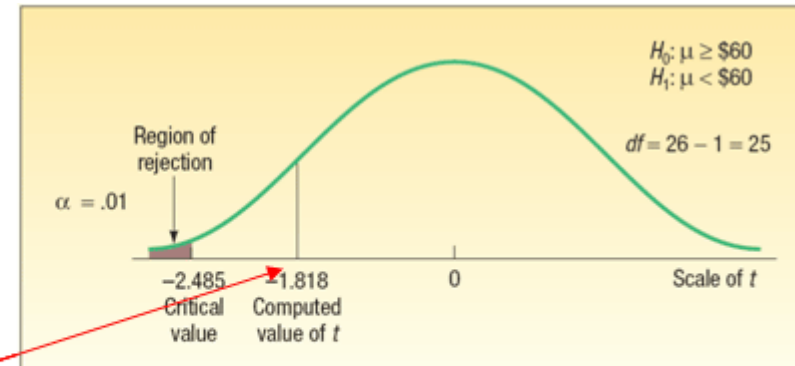


# Testing for a Population Mean with a Known Population Standard Deviation- Example

## Step 4: Formulate the decision rule.

Reject  $H_0$  if  $t < -t_{\alpha, n-1}$

$$\begin{aligned} t &< -t_{\alpha, n-1} \\ \frac{\bar{X} - \mu}{s / \sqrt{n}} &< -t_{\alpha, n-1} \\ \frac{\$56.42 - \$60}{\$10.04 / \sqrt{26}} &< -t_{.01, 26-1} \\ -1.818 &\text{ is not } < -2.485 \end{aligned}$$



## Step 5: Make a decision and interpret the result.

Because -1.818 does not fall in the rejection region,  $H_0$  is not rejected at the .01 significance level. We have not demonstrated that the cost-cutting measures reduced the mean cost per claim to less than \$60. The difference of \$3.58 ( $\$56.42 - \$60$ ) between the sample mean and the population mean could be due to sampling error.

## Testing for a Population Mean with an Unknown Population Standard Deviation- Example

The current rate for producing 5 amp fuses at Neary Electric Co. is 250 per hour. A new machine has been purchased and installed that, according to the supplier, will increase the production rate. A sample of 10 randomly selected hours from last month revealed the mean hourly production on the new machine was 256 units, with a sample standard deviation of 6 per hour.

At the .05 significance level can Neary conclude that the new machine is faster?

Testing for a Population Mean with a  
Known Population Standard Deviation- Example *continued*

Step 1: State the null and the alternate hypothesis.

$$H_0: \mu \leq 250; \quad H_1: \mu > 250$$

Step 2: Select the level of significance.

It is .05.

Step 3: Find a test statistic. Use the t distribution because the population standard deviation is not known and the sample size is less than 30.

Testing for a Population Mean with a  
Known Population Standard Deviation- Example *continued*

Step 4: State the decision rule.

There are  $10 - 1 = 9$  degrees of freedom. The null hypothesis is rejected if  $t > 1.833$ .

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{256 - 250}{6/\sqrt{10}} = 3.162$$

Step 5: Make a decision and interpret the results.

The null hypothesis is rejected. The mean number produced is more than 250 per hour.

# Tests Concerning Proportion

- A Proportion is the fraction or percentage that indicates the part of the population or sample having a particular trait of interest.
- The sample proportion is denoted by  $p$  and is found by  $x/n$
- The test statistic is computed as follows:

**TEST OF HYPOTHESIS, ONE PROPORTION**

$$z = \frac{p - \pi}{\sqrt{\frac{\pi(1 - \pi)}{n}}} \quad [10-3]$$

where:

$\pi$  is the population proportion.

$p$  is the sample proportion.

$n$  is the sample size.



## Assumptions in Testing a Population Proportion using the z-Distribution

- A random sample is chosen from the population.
- It is assumed that the binomial assumptions discussed in Chapter 6 are met:
  - (1) the sample data collected are the result of counts;
  - (2) the outcome of an experiment is classified into one of two mutually exclusive categories—a “success” or a “failure”;
  - (3) the probability of a success is the same for each trial; and (4) the trials are independent
- The test we will conduct shortly is appropriate when both  $n\pi$  and  $n(1 - \pi)$  are at least 5.
- When the above conditions are met, the normal distribution can be used as an approximation to the binomial distribution

# Test Statistic for Testing a Single Population Proportion

Sample proportion

Hypothesized population proportion

$$z = \frac{p - \pi}{\sqrt{\frac{\pi(1 - \pi)}{n}}}$$

Sample size

The diagram shows the test statistic formula for a single population proportion. The formula is  $z = \frac{p - \pi}{\sqrt{\frac{\pi(1 - \pi)}{n}}}$ . The entire formula is enclosed in a light green rectangular box. Three arrows point from text labels to parts of the formula: 'Sample proportion' points to the  $p$  in the numerator; 'Hypothesized population proportion' points to the  $\pi$  in the numerator; and 'Sample size' points to the  $n$  in the denominator.

# Test Statistic for Testing a Single Population Proportion - Example

Suppose prior elections in a certain state indicated it is necessary for a candidate for governor to receive at least 80 percent of the vote in the northern section of the state to be elected. The incumbent governor is interested in assessing his chances of returning to office and plans to conduct a survey of 2,000 registered voters in the northern section of the state. Using the hypothesis-testing procedure, assess the governor's chances of reelection.

# Test Statistic for Testing a Single Population Proportion - Example

**Step 1: State the null hypothesis and the alternate hypothesis.**

$$H_0: \pi \geq .80$$

$$H_1: \pi < .80$$

(note: keyword in the problem "*at least*")

**Step 2: Select the level of significance.**

$$\alpha = 0.01 \text{ as stated in the problem}$$

**Step 3: Select the test statistic.**

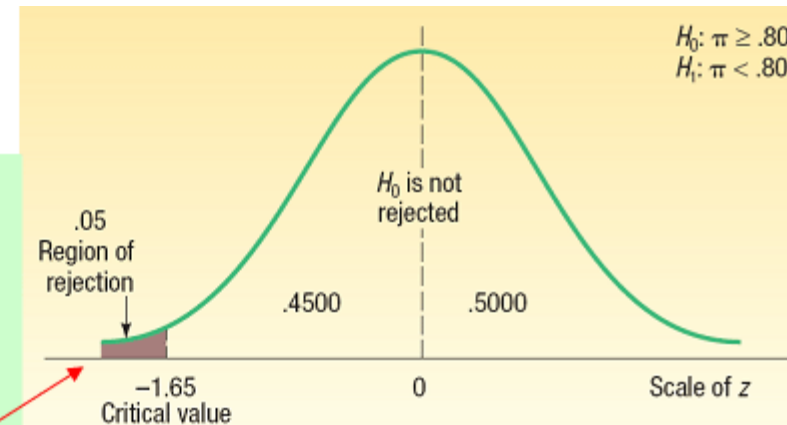
Use Z-distribution since the assumptions are met and  $n\pi$  and  $n(1-\pi) \geq 5$

# Testing for a Population Proportion - Example

**Step 4: Formulate the decision rule.**

Reject  $H_0$  if  $Z < -Z_\alpha$

$$\begin{aligned} Z &< -Z_\alpha \\ \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} &< -Z_\alpha \\ \frac{1,550}{2,000} - .80 &< -1.65 \\ \frac{\sqrt{.80(1-.80)}}{\sqrt{2,000}} &< -1.65 \\ -2.80 &< -1.65 \end{aligned}$$



**Step 5: Make a decision and interpret the result.**

The computed value of  $z$  (2.80) is in the rejection region, so the null hypothesis is rejected at the .05 level. The difference of 2.5 percentage points between the sample percent (77.5 percent) and the hypothesized population percent (80) is statistically significant. The evidence at this point does not support the claim that the incumbent governor will return to the governor's mansion for another four years.

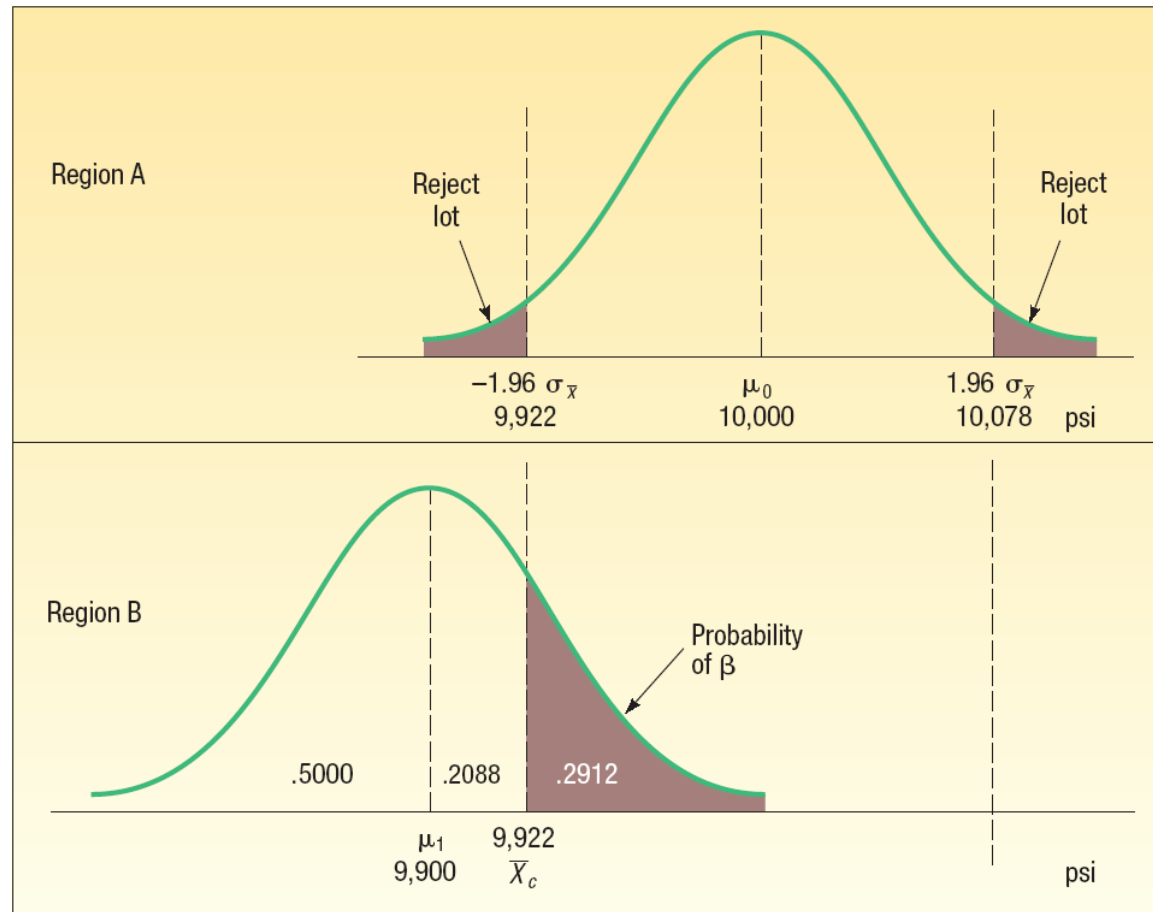
# Type II Error

- Recall **Type I Error**, the level of significance, denoted by the Greek letter “ $\alpha$ ”, is defined as the probability of rejecting the null hypothesis when it is actually true.
- **Type II Error**, denoted by the Greek letter “ $\beta$ ”, is defined as the probability of “accepting” the null hypothesis when it is actually false.

# Type II Error - Example

A manufacturer purchases steel bars to make cotter pins. Past experience indicates that the mean tensile strength of all incoming shipments is 10,000 psi and that the standard deviation,  $\sigma$ , is 400 psi. In order to make a decision about incoming shipments of steel bars, the manufacturer set up this rule for the quality-control inspector to follow: "Take a sample of 100 steel bars. At the .05 significance level if the sample mean strength falls between 9,922 psi and 10,078 psi, accept the lot. Otherwise the lot is to be rejected."

# Type I and Type II Errors Illustrated



**CHART 10-9** Charts Showing Type I and Type II Errors



# Type II Error Computed

The number of standard units ( $z$  value) between the mean of the incoming lot (9,900), designated by  $\mu_1$ , and  $\bar{X}_c$ , representing the critical value for 9,922, is computed by:

**TYPE II ERROR**

$$z = \frac{\bar{X}_c - \mu_1}{\sigma/\sqrt{n}}$$

**[10-4]**

With  $n = 100$  and  $\sigma = 400$ , the value of  $z$  is 0.55:

$$z = \frac{\bar{X}_c - \mu_1}{\sigma/\sqrt{n}} = \frac{9,922 - 9,900}{400/\sqrt{100}} = \frac{22}{40} = 0.55$$

The area under the curve between 9,900 and 9,922 (a  $z$  value of 0.55) is .2088. The area under the curve beyond 9,922 pounds is .5000 - .2088, or .2912; this is the probability of making a Type II error—that is, accepting an incoming lot of steel bars when the population mean is 9,900 psi.

# Type II Errors For Varying Mean Levels

**TABLE 10-4** Probabilities of a Type II Error for  $\mu_0 = 10,000$  Pounds and Selected Alternative Means, .05 Level of Significance

<b>Selected Alternative Mean (pounds)</b>	<b>Probability of Type II Error (<math>\beta</math>)</b>	<b>Probability of Not Making a Type II Error (<math>1 - \beta</math>)</b>
9,820	.0054	.9946
9,880	.1469	.8531
9,900	.2912	.7088
9,940	.6736	.3264
9,980	.9265	.0735
10,000	— *	—
10,020	.9265	.0735
10,060	.6736	.3264
10,100	.2912	.7088
10,120	.1469	.8531
10,180	.0054	.9946

\*It is not possible to make a Type II error when  $\mu = \mu_0$ .

End of Chapter 10

# Two-sample Tests of Hypothesis

Chapter 11



# GOALS

- Conduct a test of a hypothesis about the difference between two independent population means.
- Conduct a test of a hypothesis about the difference between two population proportions.
- Conduct a test of a hypothesis about the mean difference between paired or dependent observations.
- Understand the difference between dependent and independent samples.

# Comparing two populations – Some Examples

- Is there a difference in the mean value of residential real estate sold by male agents and female agents in south Florida?
- Is there a difference in the mean number of defects produced on the day and the afternoon shifts at Kimble Products?
- Is there a difference in the mean number of days absent between young workers (under 21 years of age) and older workers (more than 60 years of age) in the fast-food industry?
- Is there is a difference in the proportion of Ohio State University graduates and University of Cincinnati graduates who pass the state Certified Public Accountant Examination on their first attempt?
- Is there an increase in the production rate if music is piped into the production area?

# Comparing Two Population Means

- No assumptions about the shape of the populations are required.
- The samples are from independent populations.
- The formula for computing the value of  $z$  is:

Use if sample sizes  $> 30$   
or if  $\sigma_1$  and  $\sigma_2$  are known

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Use if sample sizes  $> 30$   
and if  $\sigma_1$  and  $\sigma_2$  are unknown

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

# EXAMPLE 1

The U-Scan facility was recently installed at the Byrne Road Food-Town location. The store manager would like to know if the mean checkout time using the standard checkout method is longer than using the U-Scan. She gathered the following sample information. The time is measured from when the customer enters the line until their bags are in the cart. Hence the time includes both waiting in line and checking out.

Customer Type	Sample Mean	Population Standard Deviation	Sample Size
Standard	5.50 minutes	0.40 minutes	50
U-Scan	5.30 minutes	0.30 minutes	100



# EXAMPLE 1 *continued*

Step 1: State the null and alternate hypotheses.

$$H_0: \mu_S \leq \mu_U$$

$$H_1: \mu_S > \mu_U$$

Step 2: State the level of significance.

The .01 significance level is stated in the problem.

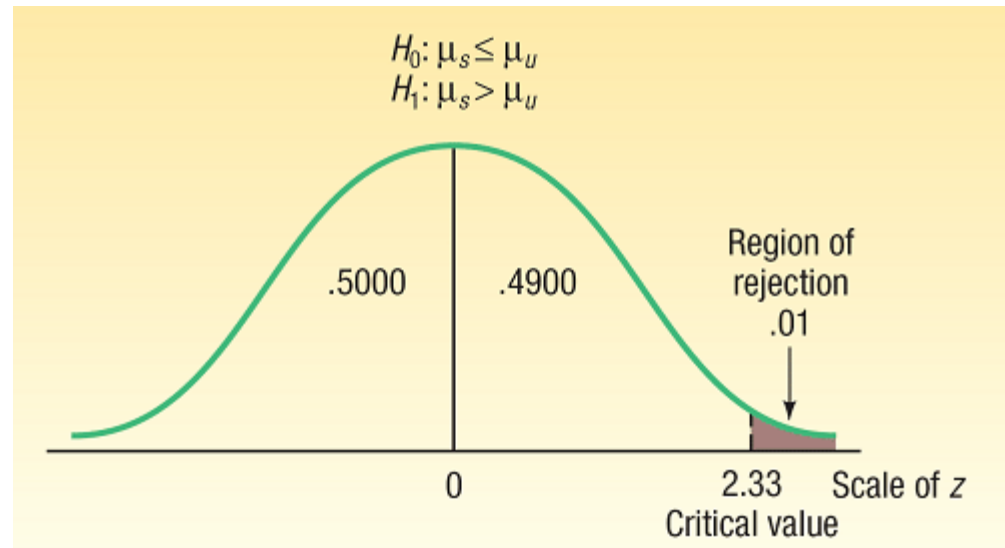
Step 3: Find the appropriate test statistic.

Because both samples are more than 30, we can use *z-distribution* as the test statistic.

# Example 1 *continued*

Step 4: State the decision rule.

$$\text{Reject } H_0 \text{ if } Z > Z_\alpha$$
$$Z > 2.33$$



# Example 1 *continued*

Step 5: Compute the value of  $z$  and make a decision

Customer Type	Sample Mean	Population Standard Deviation	Sample Size
Standard	5.50 minutes	0.40 minutes	50
U-Scan	5.30 minutes	0.30 minutes	100

$$\begin{aligned} z &= \frac{\bar{X}_s - \bar{X}_u}{\sqrt{\frac{\sigma_s^2}{n_s} + \frac{\sigma_u^2}{n_u}}} \\ &= \frac{5.5 - 5.3}{\sqrt{\frac{0.40^2}{50} + \frac{0.30^2}{100}}} \\ &= \frac{0.2}{0.064} = 3.13 \end{aligned}$$

The computed value of 3.13 is larger than the critical value of 2.33. Our decision is to reject the null hypothesis. The difference of .20 minutes between the mean checkout time using the standard method is too large to have occurred by chance. We conclude the U-Scan method is faster.

# Two-Sample Tests about Proportions

Here are several examples.

- The vice president of human resources wishes to know whether there is a difference in the proportion of hourly employees who miss more than 5 days of work per year at the Atlanta and the Houston plants.
- General Motors is considering a new design for the Pontiac Grand Am. The design is shown to a group of potential buyers under 30 years of age and another group over 60 years of age. Pontiac wishes to know whether there is a difference in the proportion of the two groups who like the new design.
- A consultant to the airline industry is investigating the fear of flying among adults. Specifically, the company wishes to know whether there is a difference in the proportion of men versus women who are fearful of flying.

# Two Sample Tests of Proportions

- We investigate whether two samples came from populations with an equal proportion of successes.
- The two samples are pooled using the following formula.

**POOLED PROPORTION**

$$p_c = \frac{X_1 + X_2}{n_1 + n_2}$$

[11-4]

where:

$X_1$  is the number possessing the trait in the first sample.

$X_2$  is the number possessing the trait in the second sample.

$n_1$  is the number of observations in the first sample.

$n_2$  is the number of observations in the second sample.

# Two Sample Tests of Proportions *continued*

The value of the test statistic is computed from the following formula.

**TWO-SAMPLE TEST  
OF PROPORTIONS**

$$z = \frac{p_1 - p_2}{\sqrt{\frac{p_c(1 - p_c)}{n_1} + \frac{p_c(1 - p_c)}{n_2}}}$$

**POOLED PROPORTION**

$$p_c = \frac{X_1 + X_2}{n_1 + n_2}$$

# Two Sample Tests of Proportions - Example

Manelli Perfume Company recently developed a new fragrance that it plans to market under the name Heavenly. A number of market studies indicate that Heavenly has very good market potential. The Sales Department at Manelli is particularly interested in whether there is a difference in the proportions of younger and older women who would purchase Heavenly if it were marketed. There are two independent populations, a population consisting of the younger women and a population consisting of the older women. Each sampled woman will be asked to smell Heavenly and indicate whether she likes the fragrance well enough to purchase a bottle.

# Two Sample Tests of Proportions - Example

Step 1: State the null and alternate hypotheses.

$$H_0: \pi_1 = \pi_2$$

$$H_1: \pi_1 \neq \pi_2$$

Step 2: State the level of significance.

The .05 significance level is stated in the problem.

Step 3: Find the appropriate test statistic.

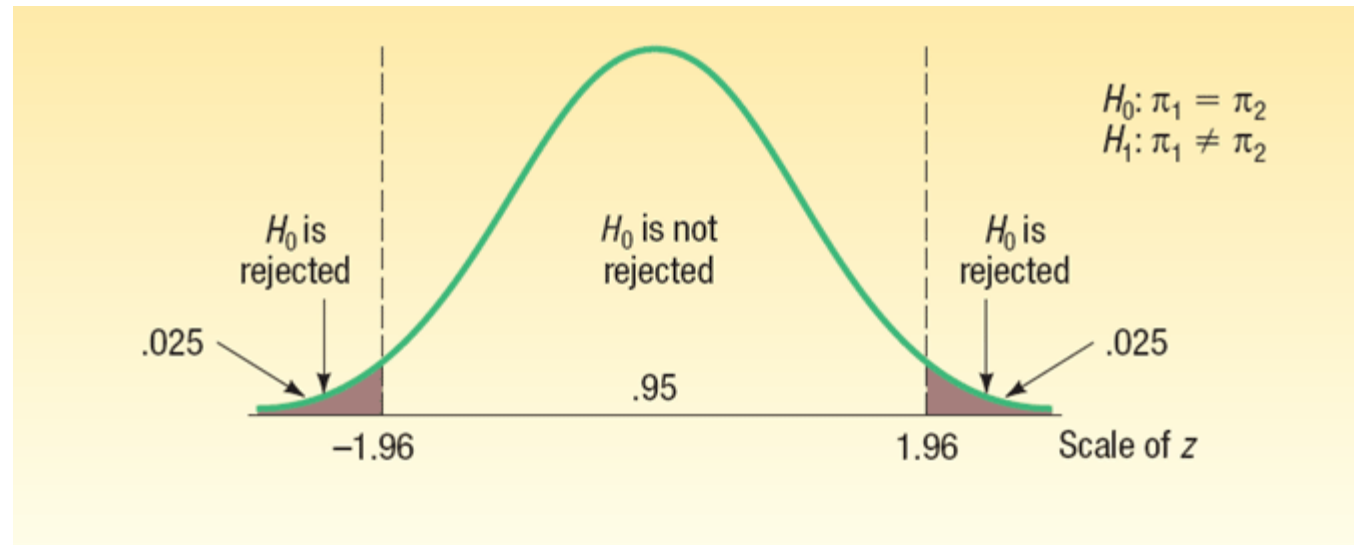
We will use the z-distribution



# Two Sample Tests of Proportions - Example

Step 4: State the decision rule.

Reject  $H_0$  if  $Z > Z_{\alpha/2}$  or  $Z < -Z_{\alpha/2}$   
 $Z > 1.96$  or  $Z < -1.96$



# Two Sample Tests of Proportions - Example

Step 5: Compute the value of  $z$  and make a decision

Let  $p_1$  = young women  $p_2$  = older women

$$p_1 = \frac{X_1}{n_1} = \frac{19}{100} = .19 \quad p_2 = \frac{X_2}{n_2} = \frac{62}{200} = .31$$

$$p_c = \frac{X_1 + X_2}{n_1 + n_2} = \frac{19 + 62}{100 + 200} = \frac{81}{300} = 0.27$$

$$z = \frac{p_1 - p_2}{\sqrt{\frac{p_c(1 - p_c)}{n_1} + \frac{p_c(1 - p_c)}{n_2}}} = \frac{.19 - .31}{\sqrt{\frac{.27(1 - .27)}{100} + \frac{.27(1 - .27)}{200}}} = -2.21$$

The computed value of 2.21 is in the area of rejection. Therefore, the null hypothesis is rejected at the .05 significance level. To put it another way, we reject the null hypothesis that the proportion of young women who would purchase Heavenly is equal to the proportion of older women who would purchase Heavenly.

# Two Sample Tests of Proportions – Example (Minitab Solution)

The screenshot displays the Minitab software interface. The main window, titled 'Session', shows the output of a 'Test and CI for Two Proportions'. The data table indicates two samples: Sample 1 with 19 successes out of 100 trials (0.190000), and Sample 2 with 62 successes out of 200 trials (0.310000). The analysis results show a difference of -0.12, a 95% confidence interval of (-0.220102, -0.0198978), and a p-value of 0.027 for the test of no difference.

```
MINITAB - Untried
File Edit Data Calc Stat Graph Editor Tools Window Help
[Icons]
Session
Test and CI for Two Proportions

Sample  X   N  Sample p
1       19  100  0.190000
2       62  200  0.310000

Difference = p (1) - p (2)
Estimate for difference: -0.12
95% CI for difference: (-0.220102, -0.0198978)
Test for difference = 0 (vs not = 0): Z = -2.21 P-Value = 0.027

Project Ma... Worksheet...
Welcome to Minitab, press F1 for help. Editable 4:20 PM
Start MINITAB - Untried Chapter11-13e - Micros... Shot11-2 - Paint Address 4:20 PM
```

## Comparing Population Means with Unknown Population Standard Deviations (the Pooled $t$ -test)

The  $t$  distribution is used as the test statistic if one or more of the samples have less than 30 observations. The required assumptions are:

1. Both populations must follow the normal distribution.
2. The populations must have equal standard deviations.
3. The samples are from independent populations.

# Small sample test of means *continued*

Finding the value of the test statistic requires two steps.

1. Pool the sample standard deviations.
2. Use the pooled standard deviation in the formula.

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{s_p^2 \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}}$$

# Comparing Population Means with Unknown Population Standard Deviations (the Pooled $t$ -test)

Owens Lawn Care, Inc., manufactures and assembles lawnmowers that are shipped to dealers throughout the United States and Canada. Two different procedures have been proposed for mounting the engine on the frame of the lawnmower. The question is: Is there a difference in the mean time to mount the engines on the frames of the lawnmowers? The first procedure was developed by longtime Owens employee Herb Welles (designated as procedure 1), and the other procedure was developed by Owens Vice President of Engineering William Atkins (designated as procedure 2). To evaluate the two methods, it was decided to conduct a time and motion study.

A sample of five employees was timed using the Welles method and six using the Atkins method. The results, in minutes, are shown on the right.

Is there a difference in the mean mounting times? Use the .10 significance level.

Welles (minutes)	Atkins (minutes)
2	3
4	7
9	5
3	8
2	4
	3

## Comparing Population Means with Unknown Population Standard Deviations (the Pooled $t$ -test) - Example

Step 1: State the null and alternate hypotheses.

$$H_0: \mu_1 = \mu_2$$

$$H_1: \mu_1 \neq \mu_2$$

Step 2: State the level of significance. The .10 significance level is stated in the problem.

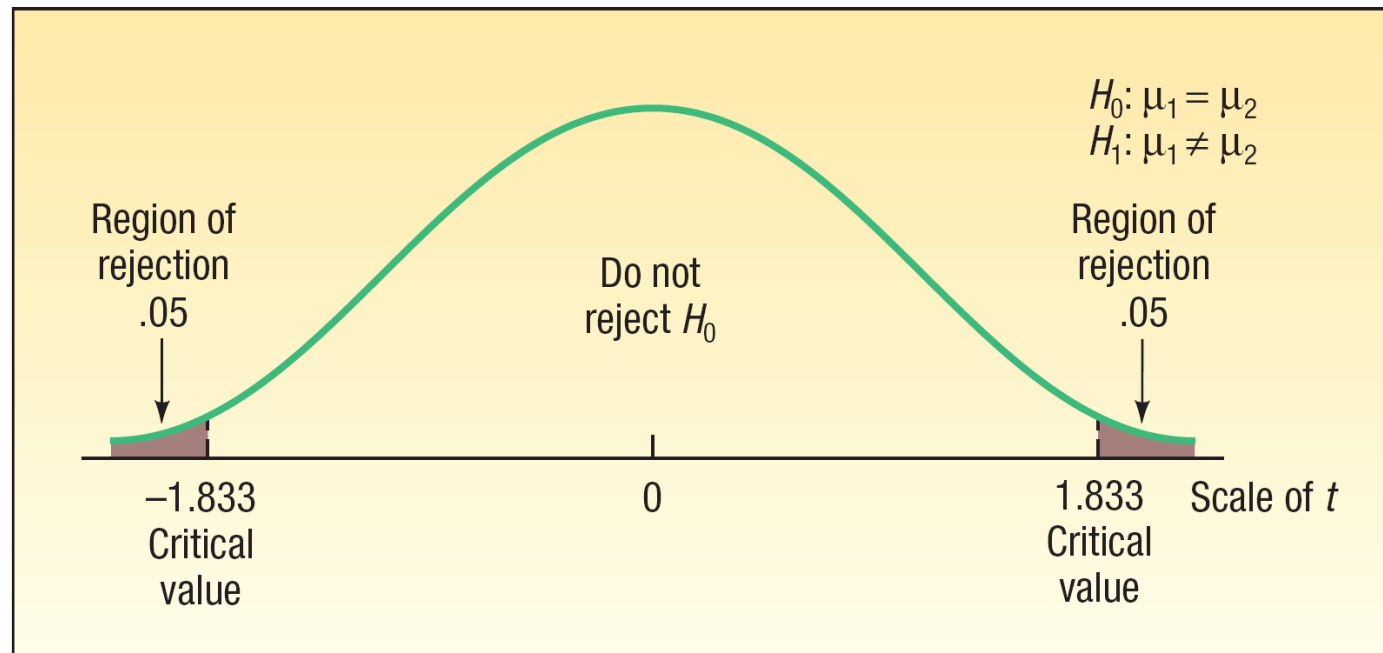
Step 3: Find the appropriate test statistic.

Because the population standard deviations are not known but are assumed to be equal, we use the pooled  $t$ -test.

# Comparing Population Means with Unknown Population Standard Deviations (the Pooled $t$ -test) - Example

Step 4: State the decision rule.

$$\begin{aligned} \text{Reject } H_0 \text{ if } & t > t_{\alpha/2, n_1+n_2-2} \text{ or } t < -t_{\alpha/2, n_1+n_2-2} \\ & t > t_{.05, 9} \text{ or } t < -t_{.05, 9} \\ & t > 1.833 \text{ or } t < -1.833 \end{aligned}$$





## Comparing Population Means with Unknown Population Standard Deviations (the Pooled $t$ -test) - Example

Step 5: Compute the value of  $t$  and make a decision

(a) Calculate the sample standard deviations

Welles Method		Atkins Method	
$X_1$	$(X_1 - \bar{X}_1)^2$	$X_2$	$(X_2 - \bar{X}_2)^2$
2	$(2 - 4)^2 = 4$	3	$(3 - 5)^2 = 4$
4	$(4 - 4)^2 = 0$	7	$(7 - 5)^2 = 4$
9	$(9 - 4)^2 = 25$	5	$(5 - 5)^2 = 0$
3	$(3 - 4)^2 = 1$	8	$(8 - 5)^2 = 9$
2	$(2 - 4)^2 = 4$	4	$(4 - 5)^2 = 1$
$\frac{20}{5}$	$\frac{34}{5}$	$\frac{30}{6}$	$\frac{22}{6}$

$$\bar{X}_1 = \frac{\sum X_1}{n_1} = \frac{20}{5} = 4 \qquad \bar{X}_2 = \frac{\sum X_2}{n_2} = \frac{30}{6} = 5$$

$$s_1 = \sqrt{\frac{\sum (X_1 - \bar{X}_1)^2}{n_1 - 1}} = \sqrt{\frac{34}{5 - 1}} = 2.9155 \qquad s_2 = \sqrt{\frac{\sum (X_2 - \bar{X}_2)^2}{n_2 - 1}} = \sqrt{\frac{22}{6 - 1}} = 2.0976$$

(b) Calculate the **pooled** sample standard deviation

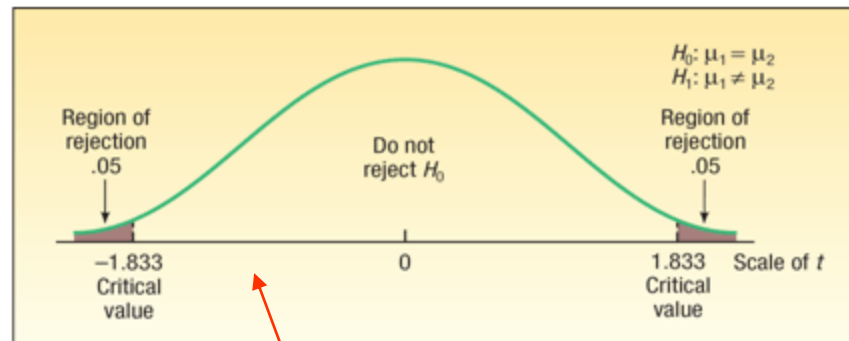
$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{(5 - 1)(2.9155)^2 + (6 - 1)(2.0976)^2}{5 + 6 - 2} = 6.2222$$

## Comparing Population Means with Unknown Population Standard Deviations (the Pooled $t$ -test) - Example

Step 5: Compute the value of  $t$  and make a decision

(c) Determine the value of  $t$

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{s_p^2 \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}} = \frac{4.00 - 5.00}{\sqrt{6.2222 \left( \frac{1}{5} + \frac{1}{6} \right)}} = -0.662$$



-0.662

The decision is not to reject the null hypothesis, because 0.662 falls in the region between -1.833 and 1.833.

We conclude that there is **no difference** in the mean times to mount the engine on the frame using the two methods.

# Comparing Population Means with Unknown Population Standard Deviations (the Pooled $t$ -test) - Example

The screenshot shows a Microsoft Excel spreadsheet with the following data and results:

	A	B	D	E	F
1	Welles	Atkins			
2	2	3	<b>t-Test: Two-Sample Assuming Equal Variances</b>		
3	4	7			
4	9	5		One	Two
5	3	8	Mean	4	5
6	2	4	Variance	8.5	4.4
7		3	Observations	5	6
8			Pooled Variance	6.222	
9			Hypothesized Mean Difference	0	
10			df	9	
11			t Stat	-0.662	
12			P(T<=t) one-tail	0.262	
13			t Critical one-tail	1.833	
14			P(T<=t) two-tail	0.525	
15			t Critical two-tail	2.262	

# Comparing Population Means with Unequal Population Standard Deviations

If it is not reasonable to assume the population standard deviations are equal, then we compute the  $t$ -statistic shown on the right.

The sample standard deviations  $s_1$  and  $s_2$  are used in place of the respective population standard deviations.

In addition, the degrees of freedom are adjusted downward by a rather complex approximation formula. The effect is to reduce the number of degrees of freedom in the test, which will require a larger value of the test statistic to reject the null hypothesis.

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$df = \frac{[(s_1^2/n_1) + (s_2^2/n_2)]^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

## Comparing Population Means with Unequal Population Standard Deviations - Example

Personnel in a consumer testing laboratory are evaluating the absorbency of paper towels. They wish to compare a set of store brand towels to a similar group of name brand ones. For each brand they dip a ply of the paper into a tub of fluid, allow the paper to drain back into the vat for two minutes, and then evaluate the amount of liquid the paper has taken up from the vat. A random sample of 9 store brand paper towels absorbed the following amounts of liquid in milliliters.

8 8 3 1 9 7 5 5 12

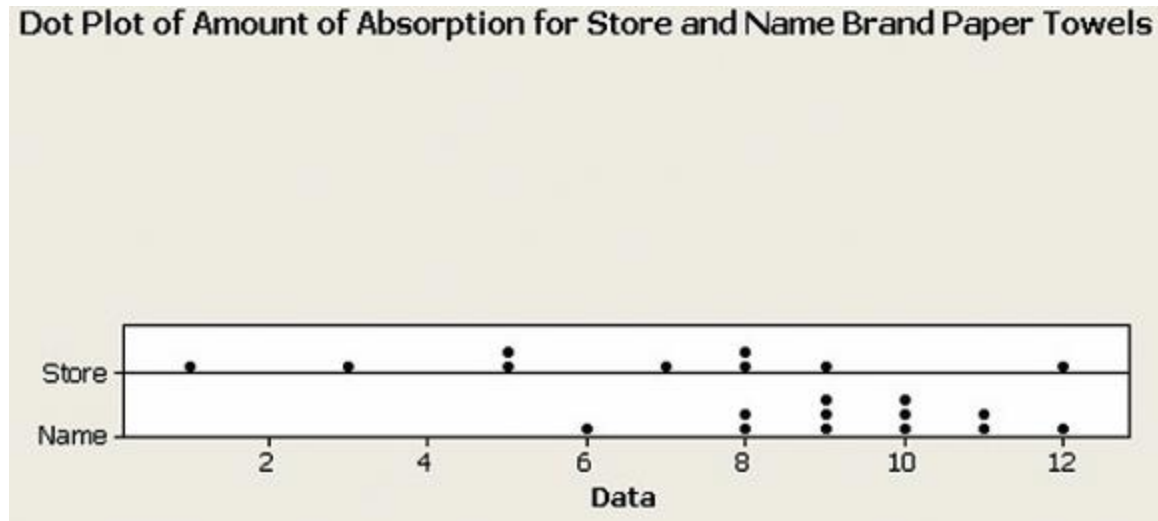
An independent random sample of 12 name brand towels absorbed the following amounts of liquid in milliliters:

12 11 10 6 8 9 9 10 11 9 8 10

Use the .10 significance level and test if there is a [difference](#) in the mean amount of liquid absorbed by the two types of paper towels.

# Comparing Population Means with Unequal Population Standard Deviations - Example

The following dot plot provided by MINITAB shows the variances to be unequal.



# Comparing Population Means with Unequal Population Standard Deviations - Example

Step 1: State the null and alternate hypotheses.

$$H_0: \mu_1 = \mu_2$$

$$H_1: \mu_1 \neq \mu_2$$

Step 2: State the level of significance.

The .10 significance level is stated in the problem.

Step 3: Find the appropriate test statistic.

We will use unequal variances  $t$ -test

# Comparing Population Means with Unequal Population Standard Deviations - Example

Step 4: State the decision rule.

Reject  $H_0$  if

$$t > t_{\alpha/2, d.f.} \text{ or } t < -t_{\alpha/2, d.f.}$$

$$t > t_{.05, 10} \text{ or } t < -t_{.05, 10}$$

$$t > 1.812 \text{ or } t < -1.812$$

Step 5: Compute the value of  $t$  and make a decision

$$df = \frac{[(s_1^2/n_1) + (s_2^2/n_2)]^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}} = \frac{[(3.32^2/9) + (1.621^2/12)]^2}{\frac{(3.32^2/9)^2}{9 - 1} + \frac{(1.621^2/12)^2}{12 - 1}} = \frac{1.4436^2}{.1875 + .0043} = 10.88$$

## Descriptive Statistics: Store, Name

Variable	N	Mean	StDev
Store	9	6.44	3.32
Name	12	9.417	1.621

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{6.44 - 9.417}{\sqrt{\frac{3.32^2}{9} + \frac{1.621^2}{12}}} = -2.478$$

The computed value of  $t$  is less than the lower critical value, so our decision is to reject the null hypothesis. We conclude that the mean absorption rate for the two towels is not the same.



# Minitab

The screenshot displays the Minitab software interface. The main window shows a data table with columns C1 (Store) and C2 (Name). A 'Session' window is open, displaying the results of a Two-Sample T-Test and CI for Store vs Name. The output includes sample sizes, means, standard deviations, and standard errors for both groups, along with the difference estimate, 95% confidence interval, and T-test results.

	C1	C2
1	8	12
2	8	11
3	3	10
4	1	6
5	9	8
6	7	9
7	5	9
8	5	10
9	12	11
10		9
11		8
12		10
13		
14		
15		
16		
17		

**Two-Sample T-Test and CI: Store, Name**  
Two-sample T for Store vs Name

	N	Mean	StDev	SE Mean
Store	9	6.44	3.32	1.1
Name	12	9.42	1.62	0.47

Difference =  $\mu$  (Store) -  $\mu$  (Name)  
Estimate for difference: -2.97222  
95% CI for difference: (-5.65005, -0.29440)  
T-Test of difference = 0 (vs not =): T-Value = -2.47 P-Value = 0.033 DF = 10

# Two-Sample Tests of Hypothesis: Dependent Samples

Dependent samples are samples that are paired or related in some fashion.

For example:

- If you wished to buy a car you would look at the *same* car at two (or more) *different* dealerships and compare the prices.
- If you wished to measure the effectiveness of a new diet you would weigh the dieters at the start and at the finish of the program.

# Hypothesis Testing Involving Paired Observations

Use the following test when the samples are **dependent**:

$$t = \frac{\bar{d}}{s_d / \sqrt{n}}$$

Where

$\bar{d}$  is the mean of the differences

$s_d$  is the standard deviation of the differences

$n$  is the number of pairs (differences)

# Hypothesis Testing Involving Paired Observations - Example

Nickel Savings and Loan wishes to compare the two companies it uses to appraise the value of residential homes. Nickel Savings selected a sample of 10 residential properties and scheduled both firms for an appraisal. The results, reported in \$000, are shown on the table (right).

At the .05 significance level, can we conclude there is a difference in the mean appraised values of the homes?

Home	Schadek	Bowyer
1	235	228
2	210	205
3	231	219
4	242	240
5	205	198
6	230	223
7	231	227
8	210	215
9	225	222
10	249	245

# Hypothesis Testing Involving Paired Observations - Example

Step 1: State the null and alternate hypotheses.

$$H_0: \mu_d = 0$$

$$H_1: \mu_d \neq 0$$

Step 2: State the level of significance.

The .05 significance level is stated in the problem.

Step 3: Find the appropriate test statistic.

We will use the  $t$ -test

# Hypothesis Testing Involving Paired Observations - Example

Step 4: State the decision rule.

Reject  $H_0$  if

$$t > t_{\alpha/2, n-1} \text{ or } t < -t_{\alpha/2, n-1}$$

$$t > t_{.025, 9} \text{ or } t < -t_{.025, 9}$$

$$t > 2.262 \text{ or } t < -2.262$$

**TABLE 11-2** A Portion of the  $t$  Distribution from Appendix B.2

Confidence Intervals						
	80%	90%	95%	98%	99%	99.9%
df	Level of Significance for One-Tailed Test					
	0.100	0.050	0.025	0.010	0.005	0.0005
	Level of Significance for Two-Tailed Test					
	0.20	0.10	0.05	0.02	0.01	0.001
1	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.599
3	1.638	2.353	3.182	4.541	5.841	12.924
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.869
6	1.440	1.943	2.447	3.143	3.707	5.959
7	1.415	1.895	2.365	2.998	3.499	5.408
8	1.397	1.860	2.306	2.896	3.355	5.041
9	1.383	1.833	2.262	2.821	3.250	4.781
10	1.372	1.812	2.228	2.764	3.169	4.587

# Hypothesis Testing Involving Paired Observations - Example

Step 5: Compute the value of  $t$  and make a decision

Home	Schadek	Bowyer	Difference, $d$	$(d - \bar{d})$	$(d - \bar{d})^2$
1	235	228	7	2.4	5.76
2	210	205	5	0.4	0.16
3	231	219	12	7.4	54.76
4	242	240	2	-2.6	6.76
5	205	198	7	2.4	5.76
6	230	223	7	2.4	5.76
7	231	227	4	-0.6	0.36
8	210	215	-5	-9.6	92.16
9	225	222	3	-1.6	2.56
10	249	245	<u>4</u>	<u>-0.6</u>	<u>0.36</u>
			46	0	174.40

$$\bar{d} = \frac{\sum d}{n} = \frac{46}{10} = 4.60$$

$$s_d = \sqrt{\frac{\sum (d - \bar{d})^2}{n - 1}} = \sqrt{\frac{174.4}{10 - 1}} = 4.402$$

$$t = \frac{\bar{d}}{s_d / \sqrt{n}} = \frac{4.6}{4.402 / \sqrt{10}} = 3.305$$

The computed value of  $t$  is greater than the higher critical value, so our decision is to reject the null hypothesis. We conclude that **there is a difference** in the mean appraised values of the homes.





End of Chapter 11

# Analysis of Variance

## Chapter 12



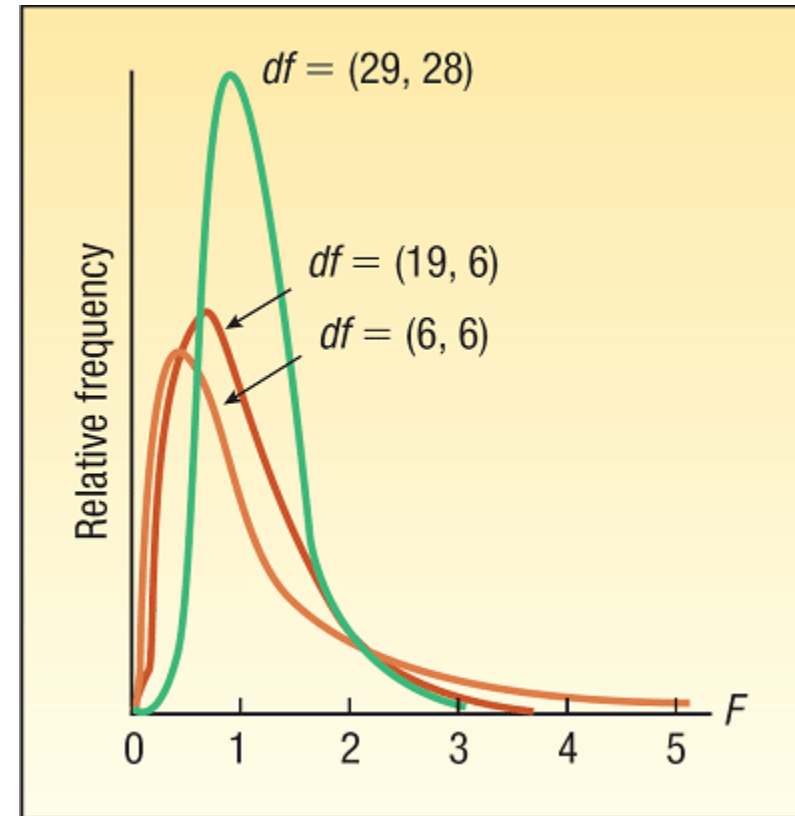
# GOALS

- List the characteristics of the F distribution.
- Conduct a test of hypothesis to determine whether the variances of two populations are equal.
- Discuss the general idea of analysis of variance.
- Organize data into a one-way and a two-way ANOVA table.
- Conduct a test of hypothesis among three or more treatment means.
- Develop confidence intervals for the difference in treatment means.
- Conduct a test of hypothesis among treatment means using a blocking variable.
- Conduct a two-way ANOVA with interaction.

# Characteristics of $F$ -Distribution

- There is a “family” of  $F$  Distributions.
- Each member of the family is determined by two parameters: the **numerator degrees of freedom** and the **denominator degrees of freedom**.
- $F$  cannot be negative, and it is a continuous distribution.
- The  $F$  distribution is positively skewed.
- Its values range from 0 to  $\infty$
- As  $F \rightarrow \infty$  the curve approaches the  $X$ -axis.

$$f(F) = \frac{\Gamma\left(\frac{\nu_1 + \nu_2}{2}\right) \nu_1^{\nu_1/2} \nu_2^{\nu_2/2} F^{\frac{\nu_1}{2}-1}}{\Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right) (\nu_1 F + \nu_2)^{\frac{\nu_1 + \nu_2}{2}}}$$



# Comparing Two Population Variances

The  $F$  distribution is used to test the hypothesis that the variance of one normal population equals the variance of another normal population. The following examples will show the use of the test:

- Two Barth shearing machines are set to produce steel bars of the same length. The bars, therefore, should have the same mean length. We want to ensure that in addition to having the same mean length they also have similar variation.
- The mean rate of return on two types of common stock may be the same, but there may be more variation in the rate of return in one than the other. A sample of 10 technology and 10 utility stocks shows the same mean rate of return, but there is likely more variation in the Internet stocks.
- A study by the marketing department for a large newspaper found that men and women spent about the same amount of time per day reading the paper. However, the same report indicated there was nearly twice as much variation in time spent per day among the men than the women.

# Test for Equal Variances

$$H_0: \sigma_1^2 = \sigma_2^2$$

$$H_1: \sigma_1^2 \neq \sigma_2^2$$

To conduct the test, we select a random sample of  $n_1$  observations from one population, and a random sample of  $n_2$  observations from the second population. The test statistic is defined as follows.

**TEST STATISTIC FOR COMPARING  
TWO VARIANCES**

$$F = \frac{S_1^2}{S_2^2}$$

**[12-1]**

# Test for Equal Variances - Example

Lammers Limos offers limousine service from the city hall in Toledo, Ohio, to Metro Airport in Detroit. Sean Lammers, president of the company, is considering two routes. One is via U.S. 25 and the other via I-75. He wants to study the time it takes to drive to the airport using each route and then compare the results. He collected the following sample data, which is reported in minutes.

Using the .10 significance level, is there a **difference** in the **variation** in the driving times for the two routes?

# Test for Equal Variances - Example

Step 1: The hypotheses are:

$$H_0: \sigma_1^2 = \sigma_2^2$$

$$H_1: \sigma_1^2 \neq \sigma_2^2$$

Step 2: The significance level is .05.

Step 3: The test statistic is the  $F$  distribution.



# Test for Equal Variances - Example

Step 4: State the decision rule.

$$\text{Reject } H_0 \text{ if } F > F_{\alpha/2, v_1, v_2}$$
$$F > F_{.05/2, 7-1, 8-1}$$
$$F > F_{.025, 6, 7}$$

TABLE 12-1 Critical Values of the  $F$  Distribution,  $\alpha = .05$

Degrees of Freedom for Denominator	Degrees of Freedom for Numerator			
	5	6	7	8
1	230	234	237	239
2	19.3	19.3	19.4	19.4
3	9.01	8.94	8.89	8.85
4	6.26	6.16	6.09	6.04
5	5.05	4.95	4.88	4.82
6	4.39	4.28	4.21	4.15
7	3.97	3.87	3.79	3.73
8	3.69	3.58	3.50	3.44
9	3.48	3.37	3.29	3.23
10	3.33	3.22	3.14	3.07

# Test for Equal Variances - Example

Step 5: Compute the value of  $F$  and make a decision

U.S. Route 25

$$\bar{X} = \frac{\sum X}{n} = \frac{408}{7} = 58.29 \quad s = \sqrt{\frac{\sum(X - \bar{X})^2}{n - 1}} = \sqrt{\frac{485.43}{7 - 1}} = 8.9947$$

Interstate 75

$$\bar{X} = \frac{\sum X}{n} = \frac{472}{8} = 59.00 \quad s = \sqrt{\frac{\sum(X - \bar{X})^2}{n - 1}} = \sqrt{\frac{134}{8 - 1}} = 4.3753$$

$$F = \frac{s_1^2}{s_2^2} = \frac{(8.9947)^2}{(4.3753)^2} = 4.23$$

The decision is to **reject the null hypothesis**, because the computed  $F$  value (4.23) is larger than the critical value (3.87).

We conclude that **there is a difference in the variation** of the travel times along the two routes.

# Test for Equal Variances – Excel Example

The screenshot shows an Excel spreadsheet with the following data and calculations:

	A	B	C	D	E	F	G	H	I
1	U. S. 25	Interstate 75		<b>F-Test Two-Sample for Variances</b>					
2	52	59							
3	67	60							
4	58	81							
5	45	51							
6	70	56							
7	54	83							
8	64	57							
9		85							
10									
11									
12									
13									
14									
15									
16									
17									
18									
19									
20									

		U. S. 25	Interstate 75
Mean		58.2857	59.0000
Variance		80.9048	19.1429
Observations		7.0000	8.0000
df		6.0000	7.0000
<b>F</b>		<b>4.2264</b>	
P(F<=f) one-tail		0.0404	
F Critical one-tail		3.8660	

# Comparing Means of Two or More Populations

- The  $F$  distribution is also used for testing whether two or more sample means came from the same or equal populations.
- Assumptions:
  - The sampled populations follow the normal distribution.
  - The populations have equal standard deviations.
  - The samples are randomly selected and are independent.

# Comparing Means of Two or More Populations

- The **Null Hypothesis** is that the population means are the same. The **Alternative Hypothesis** is that at least one of the means is different.
- The **Test Statistic** is the  $F$  distribution.
- The **Decision rule** is to reject the null hypothesis if  $F$  (computed) is greater than  $F$  (table) with numerator and denominator degrees of freedom.
- Hypothesis Setup and Decision Rule:
  - $H_0: \mu_1 = \mu_2 = \dots = \mu_k$
  - $H_1: \text{The means are not all equal}$
  - Reject  $H_0$  if  $F > F_{\alpha, k-1, n-k}$

## Analysis of Variance – F statistic

- If there are  $k$  populations being sampled, the numerator degrees of freedom is  $k - 1$ .
- If there are a total of  $n$  observations the denominator degrees of freedom is  $n - k$ .
- The test statistic is computed by:

$$F = \frac{SST/(k - 1)}{SSE/(n - k)}$$

# Comparing Means of Two or More Populations

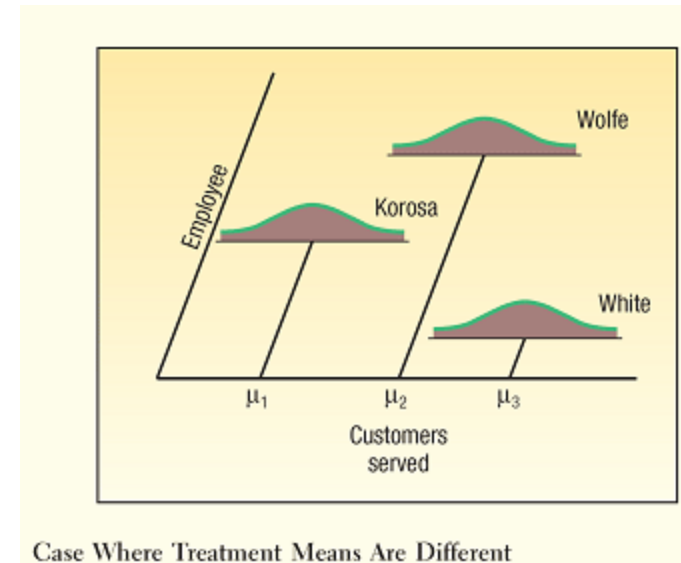
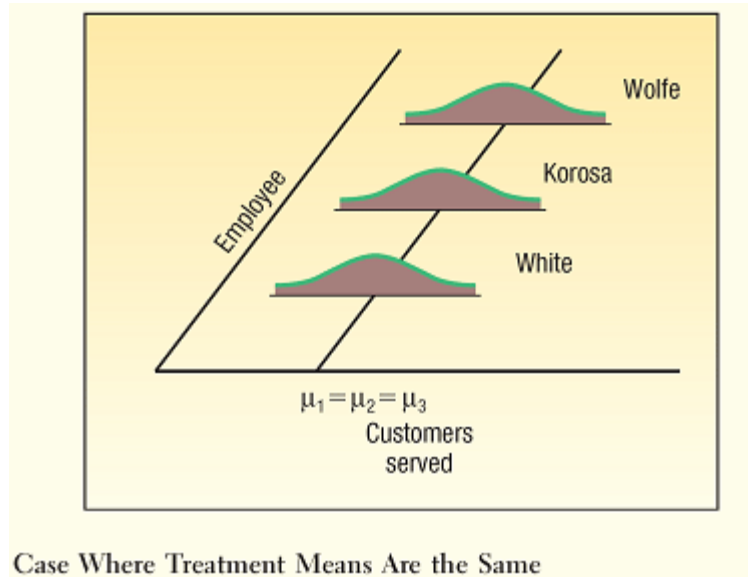
## – Illustrative Example

Joyce Kuhlman manages a regional financial center. She wishes to compare the productivity, as measured by the number of customers served, among three employees. Four days are randomly selected and the number of customers served by each employee is recorded. The results are:

<b>Wolfe</b>	<b>White</b>	<b>Korosa</b>
55	66	47
54	76	51
59	67	46
56	71	48

# Comparing Means of Two or More Populations

## – Illustrative Example





# Comparing Means of Two or More Populations – Example

Recently a group of four major carriers joined in hiring Brunner Marketing Research, Inc., to survey recent passengers regarding their level of satisfaction with a recent flight. The survey included questions on ticketing, boarding, in-flight service, baggage handling, pilot communication, and so forth. Twenty-five questions offered a range of possible answers: excellent, good, fair, or poor. A response of excellent was given a score of 4, good a 3, fair a 2, and poor a 1. These responses were then totaled, so the total score was an indication of the satisfaction with the flight. Brunner Marketing Research, Inc., randomly selected and surveyed passengers from the four airlines.

Eastern	TWA	Allegheny	Ozark
94	75	70	68
90	68	73	70
85	77	76	72
80	83	78	65
	88	80	74
		68	65
		65	

Is there a **difference** in the mean satisfaction level among the four airlines?

Use the .01 significance level.

## Comparing Means of Two or More Populations – Example

**Step 1:** State the null and alternate hypotheses.

$$H_0: \mu_E = \mu_A = \mu_T = \mu_O$$

$H_1$ : The means are not all equal

$$\text{Reject } H_0 \text{ if } F > F_{\alpha, k-1, n-k}$$

**Step 2:** State the level of significance.

The .01 significance level is stated in the problem.

**Step 3:** Find the appropriate test statistic.

Because we are comparing means of more than two groups, use the  $F$  statistic

# Comparing Means of Two or More Populations – Example

Step 4: State the decision rule.

$$\begin{aligned} \text{Reject } H_0 \text{ if } & F > F_{\alpha, k-1, n-k} \\ & F > F_{01, 4-1, 22-4} \\ & F > F_{01, 3, 18} \\ & F > 5.801 \end{aligned}$$

# Comparing Means of Two or More Populations – Example

Step 5: Compute the value of  $F$  and make a decision

ANOVA Table				
Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	$F$
Treatments	SST	$k - 1$	$SST/(k - 1) = MST$	$MST/MSE$
Error	SSE	$n - k$	$SSE/(n - k) = MSE$	
Total	<u>SS total</u>	<u><math>n - 1</math></u>		

$$SS \text{ total} = \sum(X - \bar{X}_G)^2$$

where:

$X$  is each sample observation.  
 $\bar{X}_G$  is the overall or grand mean.

$$SSE = \sum(X - \bar{X}_c)^2$$

where:

$\bar{X}_c$  is the sample mean for treatment  $c$ .

## Comparing Means of Two or More Populations – Example

$$\bar{X}_G = \frac{1,664}{22} = 75.64$$

	Eastern	TWA	Allegheny	Ozark	Total
	94	75	70	68	
	90	68	73	70	
	85	77	76	72	
	80	83	78	65	
		88	80	74	
			68	65	
			65		
Column total	349	391	510	414	1,664
<i>n</i>	4	5	7	6	22
Mean	87.25	78.20	72.86	69.00	75.64

# Computing SS Total and SSE

$$(X - \bar{X}_G)^2$$

Eastern	TWA	Allegheny	Ozark
18.36	-0.64	-5.64	-7.64
14.36	-7.64	-2.64	-5.64
9.36	1.36	0.36	-3.64
4.36	7.36	2.36	-10.64
	12.36	4.36	-1.64
		-7.64	-10.64
		-10.64	

$$(X - \bar{X}_G)^2$$

	Eastern	TWA	Allegheny	Ozark	Total
	337.09	0.41	31.81	58.37	
	206.21	58.37	6.97	31.81	
	87.61	1.85	0.13	13.25	
	19.0	54.17	5.57	113.21	
		152.77	19.01	2.69	
			58.37	113.21	
			113.21		
Total	649.91	267.57	235.07	332.54	1,485.09

$$\text{SS total} = \sum(X - \bar{X}_G)^2$$

$$(X - \bar{X}_c)^2$$

Eastern	TWA	Allegheny	Ozark
6.75	-3.2	-2.86	-1
2.75	-10.2	0.14	1
-2.25	-1.2	3.14	3
-7.25	4.8	5.14	-4
	9.8	7.14	5
		-4.86	-4
		-7.86	

$$(X - \bar{X}_c)^2$$

	Eastern	TWA	Allegheny	Ozark	Total
	45.5625	10.24	8.18	1	
	7.5625	104.04	0.02	1	
	5.0625	1.44	9.86	9	
	52.5625	23.04	26.42	16	
		96.04	50.98	25	
			23.62	16	
			61.78		
Total	110.7500	234.80	180.86	68	594.41

$$\text{SSE} = \sum(X - \bar{X}_c)^2$$

# Computing SST

$$SST = SS \text{ total} - SSE = 1,485.09 - 594.41 = 890.68.$$

ANOVA Table				
Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	<i>F</i>
Treatments	SST	$k - 1$	$SST / (k - 1) = MST$	$MST / MSE$
Error	SSE	$n - k$	$SSE / (n - k) = MSE$	
Total	$\overline{SS \text{ total}}$	$\overline{n - 1}$		

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	<i>F</i>
Treatments	890.68	3	296.89	8.99
Error	594.41	18	33.02	
Total	$\overline{1,485.09}$	$\overline{21}$		

The computed value of  $F$  is 8.99, which is greater than the critical value of 5.09, so the **null hypothesis is rejected**.

Conclusion: The population means are not all equal. The mean scores are **not the same** for the four airlines; at this point we can **only conclude there is a difference in the treatment means**. We cannot determine which treatment groups differ or how many treatment groups differ.

# Inferences About Treatment Means

- When we reject the null hypothesis that the means are equal, we may want to know which treatment means differ.
- One of the simplest procedures is through the use of confidence intervals.



## Confidence Interval for the Difference Between Two Means

$$\left(\bar{X}_1 - \bar{X}_2\right) \pm t \sqrt{MSE \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}$$

- where  $t$  is obtained from the  $t$  table with degrees of freedom  $(n - k)$ .
- $MSE = [SSE/(n - k)]$

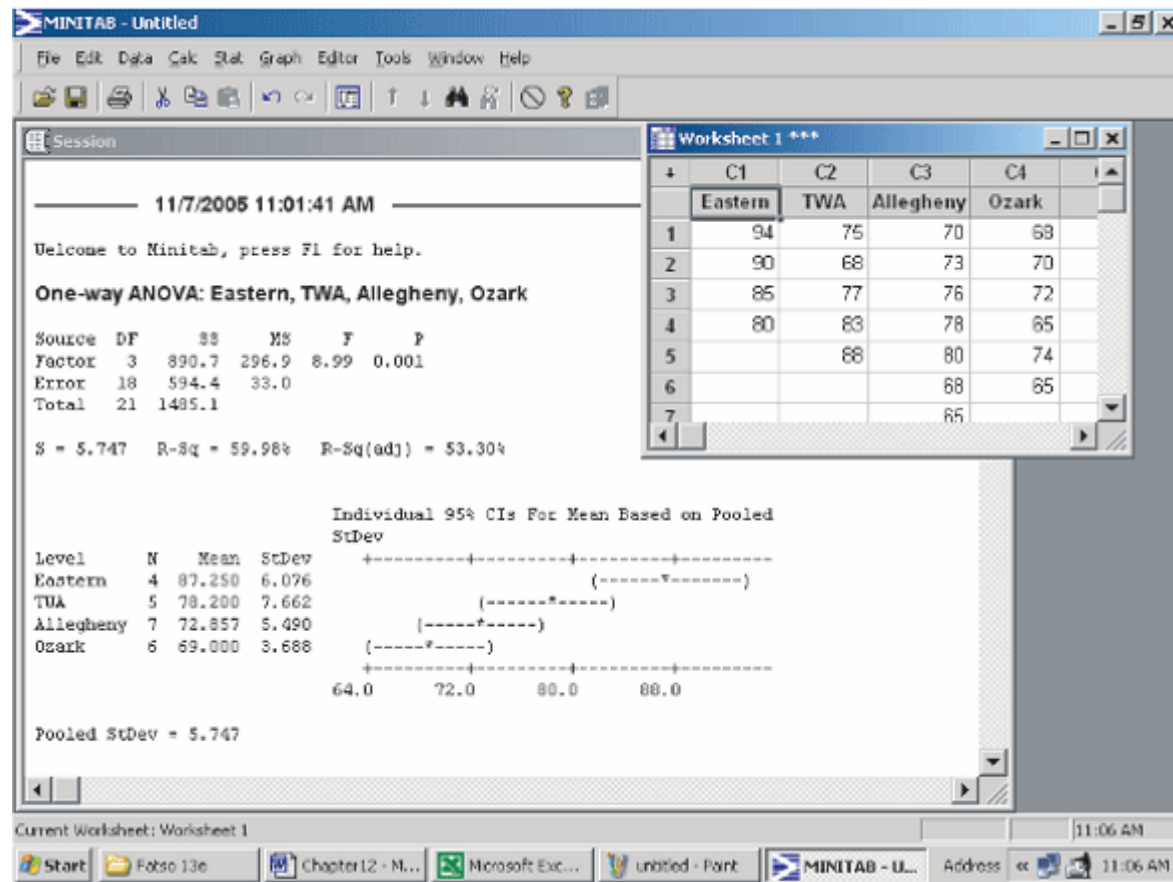
## Confidence Interval for the Difference Between Two Means - Example

From the previous example, develop a 95% confidence interval for the difference in the mean rating for Eastern and Ozark. Can we conclude that there is a difference between the two airlines' ratings?

$$\begin{aligned}(\bar{X}_E - \bar{X}_O) \pm t \sqrt{\text{MSE} \left( \frac{1}{n_E} + \frac{1}{n_O} \right)} &= (87.25 - 69.00) \pm 2.101 \sqrt{33.0 \left( \frac{1}{4} + \frac{1}{6} \right)} \\ &= 18.25 \pm 7.79\end{aligned}$$

The 95 percent confidence interval ranges from 10.46 up to 26.04. Both endpoints are positive; hence, we can conclude these treatment means differ significantly. That is, passengers on Eastern rated service significantly different from those on Ozark.

# Minitab



# Excel

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	A	B	C	D	E	F	G	H	I	J	K	
1	Eastern	TWA	Allegheny	Ozark			Anova: Single Factor					
2	94	75	70	68			SUMMARY					
3	90	68	73	70			Groups	Count	Sum	Average	Variance	
4	85	77	76	72			Eastern	4	349	87.250	36.92	
5	80	83	78	65			TWA	5	391	78.200	58.70	
6		88	80	74			Allegheny	7	510	72.857	30.14	
7			68	65			Ozark	6	414	69.000	13.60	
8			65									
9												
10												
11							ANOVA					
12							Source of Variation	SS	df	MS	F	P-value
13							Between Groups	890.68	3	296.895	8.99	0.00074
14							Within Groups	594.41	18	33.023		
15												
16							Total	1485.09	21			
17												
18												
19												
20												
21												
22												
23												

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# Two-Way Analysis of Variance

- For the two-factor ANOVA we test whether there is a significant difference between the **treatment effect** and whether there is a difference in the **blocking effect**. Let  $B_r$  be the block totals ( $r$  for rows)
- Let  $SSB$  represent the sum of squares for the blocks where:

$$SSB = \sum \left[ \frac{B_r^2}{k} \right] - \frac{(\sum X)^2}{n}$$

# Two-Way Analysis of Variance - Example

WARTA, the Warren Area Regional Transit Authority, is expanding bus service from the suburb of Starbrick into the central business district of Warren. There are four routes being considered from Starbrick to downtown Warren: (1) via U.S. 6, (2) via the West End, (3) via the Hickory Street Bridge, and (4) via Route 59.

WARTA conducted several tests to determine whether there was a difference in the mean travel times along the four routes. Because there will be many different drivers, the test was set up so each driver drove along each of the four routes. Next slide shows the travel time, in minutes, for each driver-route combination. At the .05 significance level, is there a difference in the mean travel time along the four routes? If we remove the effect of the drivers, is there a difference in the mean travel time?

## Two-Way Analysis of Variance - Example

<b>Driver</b>	<b>Travel Time From Starbrick to Warren (minutes)</b>			
	<b>U.S. 6</b>	<b>West End</b>	<b>Hickory St.</b>	<b>Rte. 59</b>
Deans	18	17	21	22
Snaverly	16	23	23	22
Ormson	21	21	26	22
Zollaco	23	22	29	25
Filbeck	25	24	28	28

# Two-Way Analysis of Variance - Example

**Step 1:** State the null and alternate hypotheses.

$$H_0: \mu_u = \mu_w = \mu_h = \mu_r$$

$H_1$ : The means are not all equal

$$\text{Reject } H_0 \text{ if } F > F_{\alpha, k-1, n-k}$$

**Step 2:** State the level of significance.

The .05 significance level is stated in the problem.

**Step 3:** Find the appropriate test statistic.

Because we are comparing means of more than two groups, use the  $F$  statistic



# Two-Way Analysis of Variance - Example

Step 4: State the decision rule.

$$\begin{aligned} \text{Reject } H_0 \text{ if } & F > F_{\alpha, v_1, v_2} \\ & F > F_{.05, k-1, n-k} \\ & F > F_{.05, 4-1, 20-4} \\ & F > F_{.05, 3, 16} \\ & F > 2.482 \end{aligned}$$

**BLOCKING VARIABLE** A second treatment variable that when included in the ANOVA analysis will have the effect of reducing the SSE term.

Driver	Travel Time From Starbrick to Warren (minutes)			
	U.S. 6	West End	Hickory St.	Rte. 59
Deans	18	17	21	22
Snaverly	16	23	23	22
Ormson	21	21	26	22
Zollaco	23	22	29	25
Filbeck	25	24	28	28

$$SSB = k \sum (\bar{X}_b - \bar{X}_G)^2$$

where

$k$  is the number of treatments.

$b$  is the number of blocks.

$\bar{X}_b$  is the sample mean of block  $b$ .

$\bar{X}_G$  is the overall or grand mean.

$$\begin{aligned}
 SSB &= k \sum (\bar{X}_b - \bar{X}_G)^2 \\
 &= 4(19.5 - 22.8)^2 + 4(21.0 - 22.8)^2 + 4(22.5 - 22.8)^2 \\
 &\quad + 4(24.75 - 22.8)^2 + 4(26.25 - 22.8)^2
 \end{aligned}$$

**SUM OF SQUARES ERROR, TWO-WAY**

$$SSE = SS \text{ total} - SST - SSB$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	<i>F</i>
Treatments	SST	$k - 1$	$SST/(k - 1) = MST$	$MST/MSE$
Blocks	SSB	$b - 1$	$SSB/(b - 1) = MSB$	$MSB/MSE$
Error	SSE	$(k - 1)(b - 1)$	$SSE/(k - 1)(b - 1) = MSE$	
Total	<u>SS total</u>	<u><math>n - 1</math></u>		

$$SSE = SS \text{ total} - SST - SSB = 229.2 - 72.8 - 119.7 = 36.7$$

Source of Variation	(1) Sum of Squares	(2) Degrees of Freedom	(3) Mean Square (1)/(2)
Treatments	72.8	3	24.27
Blocks	119.7	4	29.93
Error	36.7	12	3.06
Total	<u>229.2</u>		

# Two-Way Analysis of Variance – Excel Example

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Driver	US 6	West End	Hickory St.	Rte. 59
Deans	18	17	21	22
Snaverly	16	23	23	22
Ormson	21	21	26	22
Zollaco	23	22	29	25
Filbeck	25	24	28	28

Travel Time From Starbrick to Warren (minutes) Anova: Single Factor

SUMMARY

Groups	Count	Sum	Average	Variance
US 6	5	103	20.6	13.3
West End	5	107	21.4	7.3
Hickory St.	5	127	25.4	11.3
Rte. 59	5	119	23.8	7.2

ANOVA

Source of Variation	SS	df	MS	F
Between Groups	72.8	3	24.26667	2.48252
Within Groups	156.4	16	9.775	
Total	229.2	19		

Treatments (Route)

Error

Sheet5 / Sheet6 / Sheet2 / no reps / 3 reps / Sheet3

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Using Excel to perform the calculations. The computed value of  $F$  is 2.482, so our decision is to **not reject the null hypothesis**. We conclude there is **no difference in the mean travel time along the four routes**. There is no reason to select one of the routes as faster than the other.

# Two-Way ANOVA with Interaction

**INTERACTION** The effect of one factor on a response variable differs depending on the value of another factor.

Interaction occurs if the combination of two factors has some effect on the variable under study, in addition to each factor alone. We refer to the variable being studied as the **response** variable.

An everyday illustration of interaction is the effect of diet and exercise on weight. It is generally agreed that a person's weight (the response variable) can be controlled with two factors, diet and exercise.

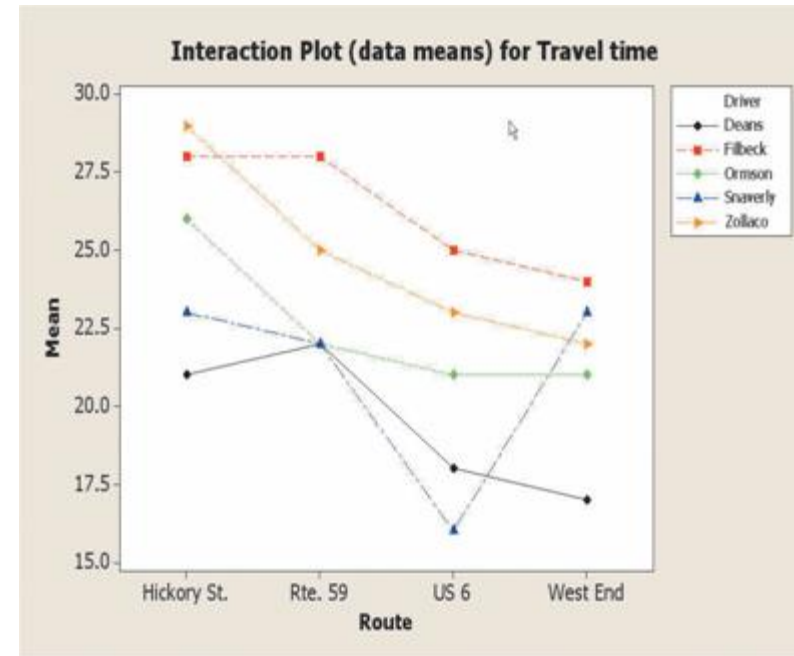
Research shows that weight is affected by diet alone and that weight is affected by exercise alone. However, the general recommended method to control weight is based on the combined or *interaction* effect of diet and exercise.

# Graphical Observation of Mean Times

Our graphical observations show us that interaction effects are possible. The next step is to conduct statistical tests of hypothesis to further investigate the possible interaction effects. In summary, our study of travel times has several questions:

- Is there really an interaction between routes and drivers?
- Are the travel times for the drivers the same?
- Are the travel times for the routes the same?

Of the three questions, we are most interested in the test for interactions. To put it another way, [does a particular route/driver combination result in significantly faster \(or slower\) driving times?](#) Also, the results of the hypothesis test for interaction affect the way we analyze the route and driver questions.



# Interaction Effect

- We can investigate these questions statistically by extending the two-way ANOVA procedure presented in the previous section. We add another source of variation, namely, the [interaction](#).
- In order to estimate the “error” sum of squares, we need [at least two](#) measurements for each driver/route combination.
- As example, suppose the experiment presented earlier is repeated by measuring two more travel times for each driver and route combination. That is, we **replicate** the experiment. Now we have three new observations for each driver/route combination.
- Using the mean of three travel times for each driver/route combination we get a more reliable measure of the mean travel time.





# Three Tests in ANOVA with Replication

The ANOVA now has three sets of hypotheses to test:

1.  $H_0$ : There is no interaction between drivers and routes.  
 $H_1$ : There is interaction between drivers and routes.
2.  $H_0$ : The driver means are the same.  
 $H_1$ : The driver means are *not* the same.
3.  $H_0$ : The route means are the same.  
 $H_1$ : The route means are *not* the same.

# ANOVA Table

Source	Sum of Squares	<i>df</i>	Mean Square	<i>F</i>
Route	Factor A	$k - 1$	$SSA / (k - 1) = MSA$	$MSA / MSE$
Driver	Factor B	$b - 1$	$SSB / (b - 1) = MSB$	$MSB / MSE$
Interaction	SSI	$(k - 1)(b - 1)$	$SSI / (k - 1)(b - 1) = MSI$	$MSI / MSE$
Error	SSE	$n - kb$	$SSE / (n - kb) = MSE$	
Total	SS total	$n - 1$		

# Excel Output

Microsoft Excel - two-way anova with interaction

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ANOVA: Two-Factor With Replication					
SUMMARY	US 6	Vest End	ickory	Siloute	5dTotal
Count	3	3	3	3	12.0
Sum	54	51	63	66	234.0
Average	18	17	21	22	19.5
Variance	9	9	1	9	9.7
Count	3	3	3	3	12.0
Sum	48	59	69	66	242.0
Average	16	20	23	22	21.0
Variance	7	7	1	4	12.7
Count	3	3	3	3	12.0
Sum	63	60	78	66	270.0
Average	21	20	26	22	22.5
Variance	4	4	7	3	7.9
Count	3	3	3	3	12.0
Sum	69	66	87	75	297.0
Average	23	22	29	25	24.8
Variance	7	4	1	1	10.2

Source of Variation	SS	df	MS	F	P-value	F crit
Sample	369.10	4	89.77	20.88	0.000	2.61
Columns	218.40	3	72.80	16.93	0.000	2.84
Interaction	110.10	12	9.18	2.13	0.036	2.00
Within	172.00	40	4.30			
Total	859.6	59				

Driver

Route

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<b>Deans: <math>H_0</math>: Route travel times are equal.</b>						<b>Snaverly: <math>H_0</math>: Route travel times are equal.</b>					
Source	DF	SS	MS	F	P	Source	DF	SS	MS	F	P
Dean RTE	3	51.00	17.00	2.43	0.140	SN RTE	3	102.00	34.00	7.16	0.012
Error	8	56.00	7.00			Error	8	38.00	4.75		
Total	11	107.00				Total	11	140.00			
<b>Ormson: <math>H_0</math>: Route travel times are equal.</b>						<b>Zollaco: <math>H_0</math>: Route travel times are equal.</b>					
Source	DF	SS	MS	F	P	Source	DF	SS	MS	F	P
Ormson RTE	3	51.00	17.00	3.78	0.059	Z-RTE	3	86.25	28.75	8.85	0.006
Error	8	36.00	4.50			Error	8	26.00	3.25		
Total	11	87.00				Total	11	112.25			
<b>Filbeck: <math>H_0</math>: Route travel times are equal.</b>											
Source	DF	SS	MS	F	P						
Filbeck RTE	3	38.25	12.75	6.38	0.016						
Error	8	16.00	2.00								
Total	11	54.25									

End of Chapter 12